Genus one mirror symmetry
(and the arithmetic Riemann-Roch theorem)

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Joint work:

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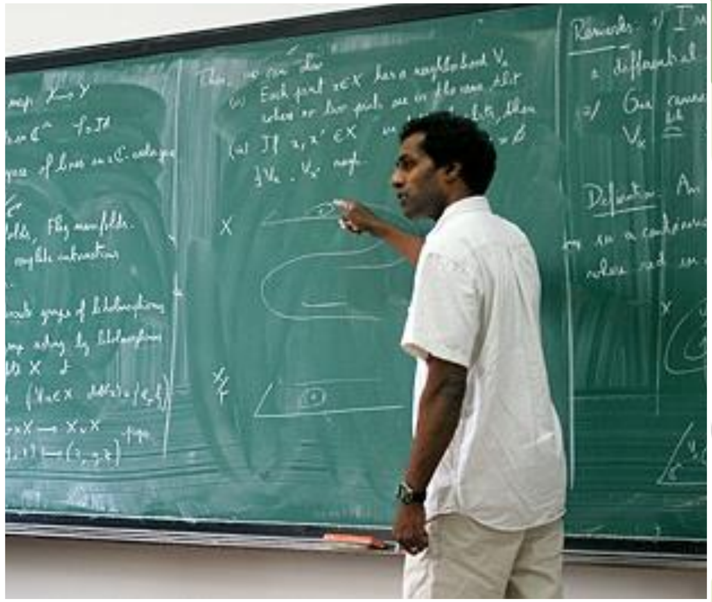

Rennes

Christophe Mourougane

On genus one mirror symmetry in higher dimensions and the BCOV conjectures, arXiv:1911.06734

BCOV invariants of Calabi-Yau manifolds and degenerations of Hodge structures, Duke Mathematical Journal Vol. -1, Issue -1 (Jan 2021)

Singularities of metrics on Hodge bundles and their topological invariants, Algebraic Geometry, Vol. 5 Issue 6 pp. 742-775

CDG (D. Eriksson, G. Freixas, C.Mourougane)

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Mirror symmetry of Bershadsky-Cecotti-Ooguri-Vafa (BCOV)

String theory formulation, emphasis on genus one.

Mirror symmetry

- Mathematical mirror symmetry is a (largely) conjectural framework working with Calabi-Yau manifolds.

- Calabi-Yau manifold: X projective manifolds of dim n admitting a non-vanishing holomorphic top-form η.

1. e.g. abelian varieties.
2. hypersurfaces of degree $n+2$ in projective space of dim $n+1$.

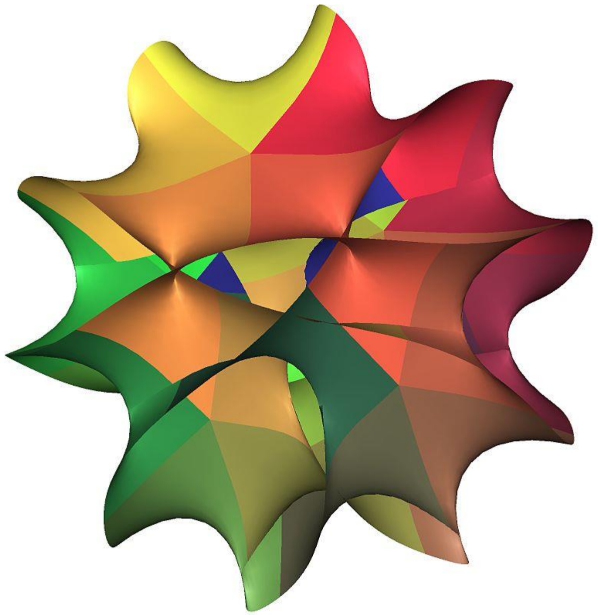
Mirror symmetry

(mirrors have mirrored Hodge diamonds)

- Should relate two types of data on "mirrored" pairs of Calabi-Yau manifolds:

- (A-side) Symplectic variations:
- Curve counting on a Calabi-Yau manifold X.
- (B-side) Holomorphic variations:
- Invariants built from a holomorphic "mirror family" of Calabi –Yau manifolds $\mathcal{X} \to D^{\times}$.

- Should relate it via a mirror map between the two sides.



Projection of Calabi-Yau of dim 3

A-side

(symplectic side)

- Fix Calabi-Yau manifold X, complexified Kähler cone:

$$H_X = H^{1,1}_{\mathbb{R}}(X)/H^{1,1}_{\mathbb{Z}}(X) + iK$$

(K = Kähler cone).

- Curve counting : Gromov-Witten invariants.
- Defined by integrating over a (virtual) fundamental class of genus g stable maps $C \to X$ landing in fixed $\beta \in H_2(X, \mathbb{Z})$.
- For hypersurfaces in projective space, $H_2(X,\mathbb{Z})$=$H_2(\mathbb{P}^{n+1},\mathbb{Z})$=$\mathbb{Z}$, and β is the degree of the image of C in projective space.

If $\dim X = 3$ or $g = 1$, the above (virtual) fundamental class is zero-dimensional, and we can count:

$$N_{g,\beta} = N_{g,\beta}(X) = \deg\left(\left[\overline{\mathcal{M}_g}(X,\beta)\right]^{virt}\right)$$

A-side

(symplectic side)

- BCOV ('94) organized this into a formal power series. Let $g = 1$ from now on.

$$N_{1,0}(X) \cdot \tau + \sum_{\beta \text{ curve class}} N_{1,\beta}(X) q^{\beta} .$$

- $q^{\beta} = \exp(2\pi i \langle \beta, \tau \rangle)$
- $\tau \in H_X$.

Denote the series by $F_{1,A}(X,\tau) = F_{1,A}(\tau)$

B-side

(holomorphic side)

B-side of the story is often interpreted as varations of Hodge structures (or complex), $\mathcal{M}_B$.

- Will often consider MUM families of CYs of dimension n:

1. $f: \mathcal{X} \to D^\times$ over the punctured multidisc $D^\times = (\mathbb{D}^\times)^d$.
2. Cannot be deformed in other directions
3. Of maximal unipotent monodromy (MUM)

The last point means that the monodromy operator has the largest possible Jordan blocks.

Informally a cusp in a moduli space of Calabi-Yau varieties:

$$D^\times \subseteq \mathcal{M}_B.$$

Mirror map: $\psi \mapsto \tau$

B $\rightarrow$ A-model

- (Morrison) In "MUM" situation, with $D^{\times} = \mathbb{D}^{\times}$, can realize "mirror map" as quotient of two (carefully picked) periods

$$\psi \mapsto \tau = \frac{\int_{\gamma_1(\psi)} \eta_\psi}{\int_{\gamma_0(\psi)} \eta_\psi}$$

- RHS should be in H_X, but it is one-dimensional in this setting and we think of it as part of $\mathbb{C}$.
- Sometimes exponentiate:

$$\psi \mapsto q = \exp(2\pi i \tau),$$

$$\mathbb{D}^{\times} \rightarrow \mathbb{D}^{\times}$$

Think of it as a natural coordinate change.

B-side, genus one, general dimension

(holomorphic side)

- BCOV ('94) predicts the existence of C^∞ real valued function, in dimension 3, $\mathfrak{F}_{1,B}$ on $D^\times$.
- $\mathfrak{F}_{1,B}$ should satisfy a type of differential equation, *the holomorphic anomaly equation*.
- $\mathfrak{F}_{1,B}$ should "know" about $F_{1,A}$ on a mirror.
- Gave a definition in terms of holomorphic analytic torsion.

(Holomorphic) Analytic torsion

- On a compact Kähler manifold (X, ω), Kodaira-Laplace operator:

$\Delta = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ acts on $A^{p,q}(X)$, with pos. eigenvalues $\Lambda_{p,q}$

$$\zeta_{p,q}(s) = \sum_{\lambda \in \Lambda_{p,q}} \frac{1}{\lambda^s},$$

- The holomorphic analytic (or holomorphic Ray-Singer) torsion:

$$T(\Omega_X^p, \omega) = \exp\left(\sum (-1)^{q+1} q\, \zeta'_{p,q}(0)\right).$$

BCOV torsion

$$\mathfrak{I}(X,\omega) := \prod T\left(\Omega_X^p,\omega\right)^{(-1)^p p}$$
$$= \exp\left(-\sum_{p,q}(-1)^{p+q}pq\zeta'_{p,q}(0)\right).$$

- Is a spectral invariant, as from notation, this *depends* on the Kähler form ω.

- BCOV:

$$\mathfrak{F}_{1,B}(X) = \frac{1}{2}\log\mathfrak{I}(X,\omega).$$

Example: One-dimensional case

$$E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$$

(mirror map is the identity)

One-dimensional case

(Description of A-side)

- $N_{1,d}(E) = topological\ coverings\ of\ degree\ d$
- $\sigma(d) = \sum_{k|d} k$, the number of subgroups of $\mathbb{Z}^2$ of index d.

$$N_{1,d}(E) = \sigma(d)/d.$$

$$F_{1,A}(q) = -\frac{1}{24}\log q + \sum \frac{\sigma(d)}{d} q^d$$
$$= \frac{-1}{24}\log \Delta$$

$$\Delta(\tau) = q\prod(1-q^n)^{24}, q = \exp(2\pi i\tau)$$

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(\tau),$$

modular form of weight 12 for $SL_2(\mathbb{Z})$

One-dimensional case

(Description of B-side)

- The $\zeta_{0,1}(s)$ for $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ (standard metric) is, up to some factor,

$$E(\tau,s)=\sum\frac{Im(\tau)^s}{|n+m\tau|^{2s}}$$

sum over $\mathbb{Z}^2 \ni (m,n) \neq (0,0)$

$$\frac{\partial}{\partial s}E(\tau,s)\Big|_{s=0}=\frac{1}{2}\log\mathfrak{T}(X,\omega)$$

One-dimensional case

(Description of B-side)

- Not difficult to derive the equation:

$$\frac{\partial^2}{\partial z\,\partial \bar{z}}\left(\frac{1}{2}\log \mathfrak{I}(E_\tau,\omega)\right) = \frac{1}{8\,\mathrm{Im}(\tau)^2}$$

(holomorphic anomaly equation in the one-dimensional case)

- Easy computation shows that all solutions of this type of equation are of the form

$$-\log\left(\sqrt{\mathrm{Im}(\tau)}\,|f|\right)$$

for a holomorphic function f on $\mathbb{H}$.

$SL_2(\mathbb{Z})$ acts by holomorphic isomorphisms on $\mathbb{H}$ which implies that $\frac{1}{2}\log \mathfrak{I}(E_\tau,\omega)$ transforms as a modular form, and implies $|f|$ transforms as a modular form.

f^{12} transforms as a modular form of weight 12.

One finds the first Kronecker limit formula:

$$\mathfrak{F}_{1,B}(E_\tau) = C + \frac{1}{2}\log \mathrm{Im}(\tau) + \frac{1}{24}\log|\Delta|$$

$$|\Delta| = \exp(-24F_{1,A}(E_\tau))$$

The mirror conjecture statement at genus one

- Suppose that we have two mirrors X (A-model) and $\mathcal{X}_\psi$ (B-model) with mirror map $\psi \mapsto \tau$.
- There is/should be a process called taking the *holomorphic limit*:

$$F_{1,B}(\tau) \coloneqq \lim_{\bar{\tau}\to\infty} \mathfrak{F}_{1,B}(\mathcal{X}_\psi)$$

- Unclear (to me) how to do this, but informally one develops $\mathfrak{F}_{1,B}(\mathcal{X}_\psi)$ in the mirror variable τ and keeps the holomorphic part.

$$F_{1,B}(\psi) = F_{1,B}(\tau) = F_{1,A}(X,\tau).$$

What is known?

- The case of dimension one was sketched.
- In dimension two, Calabi-Yau manifolds are K3 surfaces or abelian varieties. All the invariants are zero or trivial.
- For quintic threefolds this is a result of Fang-Lu-Yoshikawa.

Mathematical definition of $\mathfrak{F}_{1,B}$

- In the previous example we used a ”standard metric” on an elliptic curve.

- The ”BCOV torsion” depended on the choice of metric/Kähler form, so is not only a function in the B-model.

The BCOV invariant

- Theorem (CDG): Suppose X is a projective Calabi-Yau manifold of dimension n, with Kähler form ω. Suppose for simplicity Ricci flat of volume 1.

$$\tau_{BCOV}(X) := \frac{\mathfrak{I}(X,\omega)}{\prod_{k=0}^{n-1} Vol_{L^2}(H^k(X,\mathbb{Z}),\omega)^{(-1)^k(n-k)}}$$

- This is independent of the ω, and only depends on X.

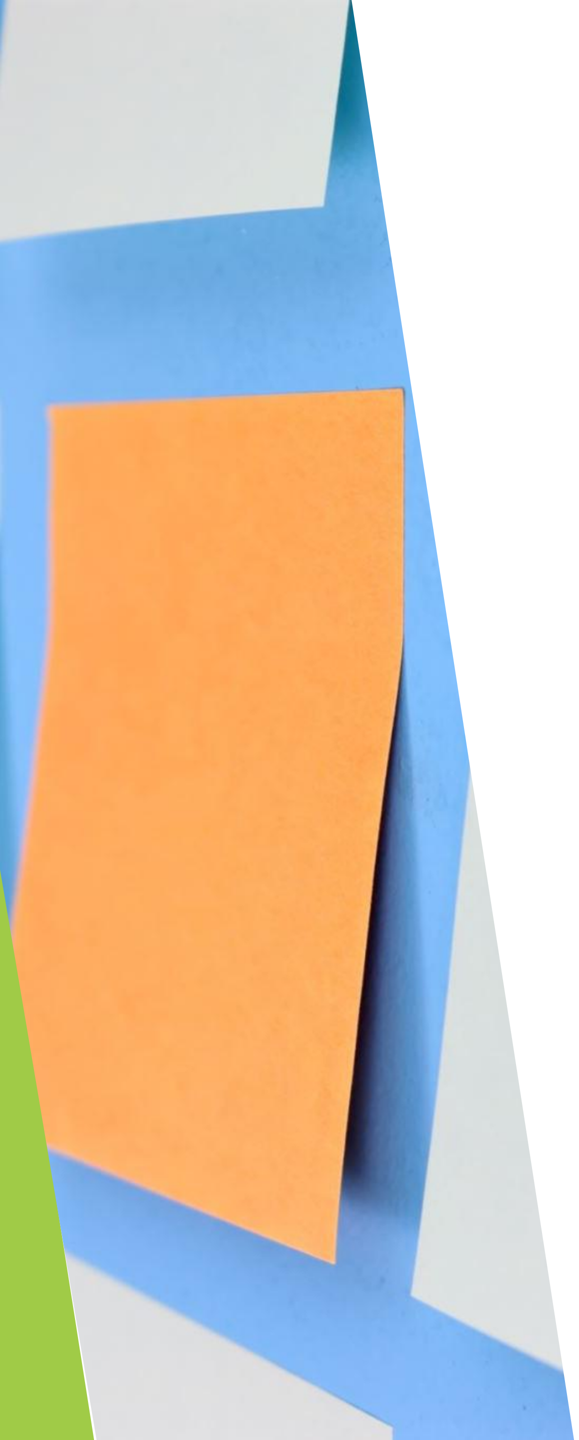
$$Vol_{L^2}(H^k(X,\mathbb{Z}),\omega) := \left(vol_{L^2}\left(H^k(X,\mathbb{R})/H^k(X,\mathbb{Z})\ \ \right)\right)^2$$

- We thus propose

$$\mathfrak{F}_{1,B} := \frac{1}{2}\log \tau_{BCOV}\,.$$

- Most known approaches to the BCOV conjecture is as follows:

1. Study boundary behavior of τ_{BCOV}.
2. Use geometry of some appropriate moduli spaces to write τ_{BCOV} in a simple way.
3. In the previous 1-dimensional case one could use the modularity to draw conclusions.



Remarks

- Call it the BCOV invariant. Generalizes known constructions in dimension 3 by Fang-Lu-Yoshikawa (2008).
- Used to prove BCOV conjecture for quintic 3-fold.
- The case $c_1(X) = 0$ was discovered recently by Yeping Zhang (2019).

Some questions about the boundary behaviour

$\mathcal{X} \to \mathbb{D}$

1-parameter proj. family of degenerating CY:s+ϵ

(CDG)

- Exists $\kappa_f \in \mathbb{Q}$, such that

$$\log \tau_{BCOV}(\mathcal{X}_\psi) = \kappa_f \log|\psi|^2 + o(\log|\psi|^2).$$

- Have general formulas for κ_f in the case of normal crossings degenerations.

$\mathcal{X} \to \mathbb{D}$
1-parameter proj. family of degenerating CY:s $+\epsilon$

(CDG)

- $\log \tau_{BCOV}(\mathcal{X}_\psi) = \kappa_f \log|\psi|^2 + o(\log|\psi|^2)$.

- For projective families of abelian varieties of dimension at least 2 and hyperkähler varieties, $\tau(\psi)$ is constant, reflecting the same behaviour on the A-side.
- In general we get topological constraints on the special fiber of such degenerations.

$\mathcal{X} \to \mathbb{D}$ 1-parameter proj. family of degenerating CY:s $+\epsilon$

(CDG)

- Suppose $f: \mathcal{X} \to \mathbb{D}$ has at worst ODP singularities at the origin.

Then,

if n is odd, $\kappa_f = \frac{n+1}{24} \# Sing(\mathcal{X}_0)$.

If n is even, $\kappa_f = \frac{2-n}{24} \# Sing(\mathcal{X}_0)$.

- If $n = 3$, this was one of the main results of Fang-Lu-Yoshikawa (2008).
- If $n = 4$, this was conjectured by Klemm-Pandharipande (2008).
- This was also conjectured by Fang-Lu-Yoshikawa for the BCOV torsion), and is related to ”universal behaviour of $\mathcal{F}_{1,B}$ close to conifold points” in the theoretical physics litterature.

Arithmetic-Riemann-Roch

A mathematical formulation of the conjecture

Grothendieck-Riemann-Roch

- $f: X \to S$ family of Calabi-Yau manifolds. Define:

$$f_* K_{X/S} = \text{direct image of relative canonical bundle}$$

$$\lambda_{BCOV}(f) \coloneqq \lambda_{BCOV} = \bigotimes_{p,q} \left(\det R^q f_* \Omega^p_{X/S}\right)^{(-1)^p p}$$

- The Grothendieck-Riemann-Roch implies that the cohomology classes of λ_{BCOV} and $\left(f_* K_{X/S}\right)^{\otimes \frac{\chi}{12}}$ are the same.
- The bundles are hence isomorphic (or some power thereof), but such an isomorphism is not unique.

Metrics

- $f : \mathcal{X} \to S$ family of complex manifolds
- $f_* K_{\mathcal{X}/S}$ carries natural L^2 −norm:

$$|\eta|^2 (\psi) := \frac{i^{n^2}}{(2\pi)^n} \int_{\mathcal{X}_\psi} \eta \wedge \overline{\eta}\,.$$

$$\lambda_{BCOV}(X) = \bigotimes_{p,q} \det H^{p,q}(X)^{(-1)^{p+q}p}$$

$H^{p,q}(X) = \ker \Delta^{p,q}$ harmonic forms. Have natural L^2 norm.

The corresponding metric on $\lambda_{BCOV}(X)$ can be modified so that it doesn't depend on an auxiliary Kähler form.

C.Soulé

H. Gillet

Arithmetic Riemann-Roch '92.

- There are metrized versions of the Grothendieck-Riemann-Roch theorem: *arithmetic Riemann-Roch theorem*, due to Gillet-Soulé.
- States that the two bundles are isometric, in a way which is unique up to multiplication by modulus one scalar. Fix this:

$$GRR: \lambda_{BCOV} \to \left(f_* K_{\mathcal{X}/S}\right)^{\otimes \frac{\chi}{12}}$$

- Will provide us with the link between BCOV formulation and our formulation.

Arithmetic Riemann-Roch theorem

(a special case of)

$$GRR: \lambda_{BCOV} \leftrightarrow \left(f_* K_{\mathcal{X}/S}\right)^{\otimes \frac{\chi}{12}}$$

- Let η be a trivializing section of $f_* K_{\mathcal{X}/S}$. Then, up to an explicit factor,

$$\tau_{BCOV}(\mathcal{X}_\psi) = \frac{|GRR(\eta)|_{L^2}^2}{|\eta|_{L^2}^{\chi/6}}.$$

In the case of elliptic curves, this relationship is known as the (first) Kronecker limit formula, responsible for the mirror symmetry statement earlier.

$$\tau_{BCOV}(\mathcal{X}_\psi) = \frac{|GRR(\eta)|_{L^2}^2}{|\eta|_{L^2}^{\chi/6}}.$$

- If η' is a trivializing section of λ_{BCOV}, define holomorphic F on S such that

$$F = \frac{GRR(\eta)}{\eta'}$$

$$\tau_{BCOV}(\mathcal{X}_\psi) = |F|^2 \frac{|\eta'|_{L^2}^2}{|\eta|_{L^2}^{\chi/6}}$$

  - η' of λ_{BCOV}
  - η of $f_* K_{\mathcal{X}/S}$
- These two sections are not given, must be constructed.

Genus one mirror symmetry formulation

- When $f: X \to \mathbb{D}^{\times}$ in the MUM situation (maximal unipotent monodromy) there are natural sections

  1. η' of λ_{BCOV}
  2. η of $f_* K_{\mathcal{X}/S}$

- Writing $\tau_{BCOV}(\mathcal{X}_\psi) = |F|^2 \frac{|\eta'|_{L^2}^2}{|\eta|_{L^2}^{\chi/6}}$, we have $|\mathrm{F}| = C\left|\exp\left((-1)^n 24 F_{1,A}(\tau(\psi))\right)\right|$.

Notice that the expression $F_{1,A}(\tau(\psi))$ involves the mirror map/coordinate.

Hypersurfaces in projective space

evidence in higher dimensions

Hypersurfaces in projective space

- Calabi-Yau hypersurface $X_{n+2} \subseteq \mathbb{P}^{n+1}$
- Interested in studying the mirror symmetry statements in this case
- For mirror quintics in $\mathbb{P}^5$, goes back to Fang-Lu-Yoshikawa in 2008.

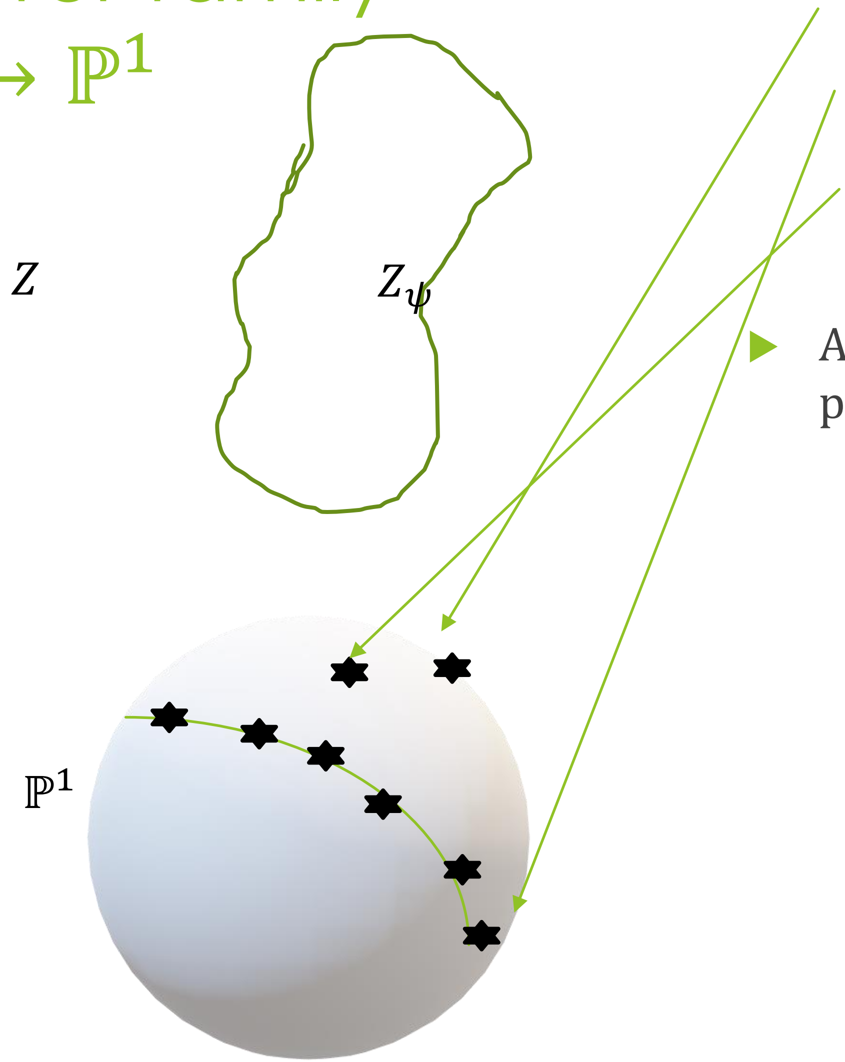
Constructing mirror family

- $X \to \mathbb{P}^1$, where for $\psi \in \mathbb{P}^1$, $X_\psi = \{F_\psi = 0\}$,

$$F_\psi = \sum x_i^{n+2} - \psi(n+2)x_0 \dots x_{n+1}$$

- The group μ_{n+2} acts on each coordinate.
- Let G be the group of such roots of unity which preserving $x_0 \dots x_{n+1}$. It acts on $X \to \mathbb{P}^1$.
- $Y_\psi = X_\psi / G$ has simple singularities, $\psi \neq \infty$, admits a (crepant) resolution $Z_\psi \to Y_\psi$.
- After desingularizing everywhere, provides a family $Z \to \mathbb{P}^1$.

Geometry of mirror family $Z \to \mathbb{P}^1$



- $\psi = \infty$, semi-stable point (MUM point).
- $\psi = \xi, \xi^{n+2} = 1$, single ODP singularity on Z_ψ.
- $\psi =$ anything else, normal smooth point.

- A neighborhood of ∞ ”is” the mirror family in the previous sense.

The mirror conjecture holds, up to constant.

- The mirror conjecture holds, up to constant, for hypersurfaces in projective space.

Strategy of proof:

1. Compute F using algebraic sections and arithmetic Riemann-Roch.
2. Modify the sections to fit the mirror symmetry setting.

Construction of the algebraic sections

$$X \to X/G \leftarrow Z$$

the cohomology of $Z_\psi \sim G$-invariant part of the cohomology of X_ψ.

- Generally more complicated, but can construct enough sections ”by hand” from $\mathbb{P}^{n+1}$, to obtain η, η'.
- Example:

$$\eta := Res\left(\frac{\psi H \Omega}{F_\psi}\right) \in H^{n,0}\left(X_\psi\right)^G,$$

$$\Omega = \sum (-1)^i x_i dx_0 \dots \widehat{dx_i} dx_{n+1}, H = x_0 \dots x_{n+1}$$

Taking derivatives (Kodaira-Spencer) produces enough sections to write down η'.

How to compute F

- Recall arithmetic Riemann-Roch: $\tau_{BCOV} = |F|^2 \frac{|\eta'|^2_{L^2\ BCOV}}{|\eta|^{\chi/6}_{L^2}}$

$$\frac{1}{2}\log \tau_{BCOV} + \frac{\chi}{12}\log|\eta|_{L^2} - \log|\eta'|_{L^2\ BCOV} = \log|F|$$

- Want to:
  - determine $F = \prod(\psi - a)^{n_a}$, since F is just a rational function on $\mathbb{P}^1$.
  - Hence, for $a \in \mathbb{P}^1$,

$$\frac{1}{2}\log \tau_{BCOV} + \frac{\chi}{12}\log|\eta|_{L^2} - \log|\eta'|_{L^2\ BCOV} = n_a \log|\psi - a| + o(\log|\psi - a|)$$

  - Since $\deg div\ F = \sum n_a = 0$, it is enough to control all points except $a = \infty$.

For $a \in \mathbb{P}^1$, want to control n_a, i.e. the asymptotic behavior close to a :

$$\frac{1}{2}\log\tau_{BCOV} + \frac{\chi}{12}\log|\eta|_{L^2} - \log|\eta'|_{L^2\ BCOV} = n_a \log|\psi - a| + o(\log|\psi - a|)$$

Outside of $ODP's$ everything is trivial.

$\log\tau_{BCOV}$ close to ODP points was recalled earlier.

Asymptotic behavior

- Theorem (CDG): Close to $a \in \mathbb{P}^1$,

- $\log|\eta'|_{L^2} = \alpha' \log|\psi - a| + o(\log|\psi - a|)$
- $\log|\eta|_{L^2} = \alpha \log|\psi - a| + o(\log|\psi - a|)$

- These can be expressed in terms of monodromy eigenvalues, and is part of a refinement of Schmid's nilpotent orbit theorem.

Conclusions

- We hence control the logarithmic behaviour of

1. $\log \tau_{BCOV}(Z_\psi)$
2. $\log|\eta'|_{L^2}, \log|\eta|_{L^2}$

around all points except $\psi = \infty$. Some delicate computations later, one finds an expression

$$F(\psi) = C\frac{\psi^{(n+2)a}}{(1-\psi^{n+2})^b}$$

Where, $C \in \mathbb{R}_{>0}$,

$$a = (-1)^n \frac{n(n+1)}{6} - \frac{\chi(Z_\psi)}{12(n+2)}$$

$$b = (-1)^n \frac{n(3n-2)}{24}$$

- The computation is based on:

1. Direct computations of limiting mixed Hodge structures,
2. Numerical tricks involving the Yukawa coupling at infinitywhich is supposed to be controlling genus 0 Gromov-Witten invariants in dimension 3.

Modify the sections

- The weight filtration on the limiting mixed Hodge structure at infinity:

$$W_0 \subseteq W_1 = W_2 \subseteq \cdots \subseteq W_{2n-1} = W_{2n} = H^n_{lim}$$

- Theorem(CDG): Fix a basis $\gamma.$ of $\left(H^{n-1}_{lim}\right)^\vee$ adapted to the weight filtration. There is a unique holomorphic basis $\tilde{\eta}.$ adapted to the Hodge filtration on $R^{n-1}f_*\mathbb{C}$ such that

1. $\int_{\gamma_k} \tilde{\eta}_p = \begin{Bmatrix} 0, k < p \\ 1, k = p \end{Bmatrix}$
2. $\tilde{\eta}.$ extends to a basis of the "canonical" extension of $R^n f_*\mathbb{C}$ in a neighborhood of ∞.

Sections adapted to the mirror situation

- The basis $\tilde{\eta}_\cdot$ is of a more transcendental nature than η, η_k.
- Only defined in a neighborhood of ∞.
- One can pass from one to the other by dividing by a lot of periods. We can fabricate periods by solving Picard-Fuchs equations and using some tricks.
- This leads to a complicated expression.
- Was related to Gromov-Witten invariants of the mirror by Zinger, '08.

Theorem (CDG)

CDG

(case in dimension 3 is due to Fang-Lu-Yoshikawa)

$$\tau_{BCOV}(Z_\psi) = \left|\exp\left((-1)^n 24 F_{1,A}(\tau(\psi))\right)\right| |\Theta|^2$$

Here, for some constant $C \in \mathbb{R}_{>0}$,

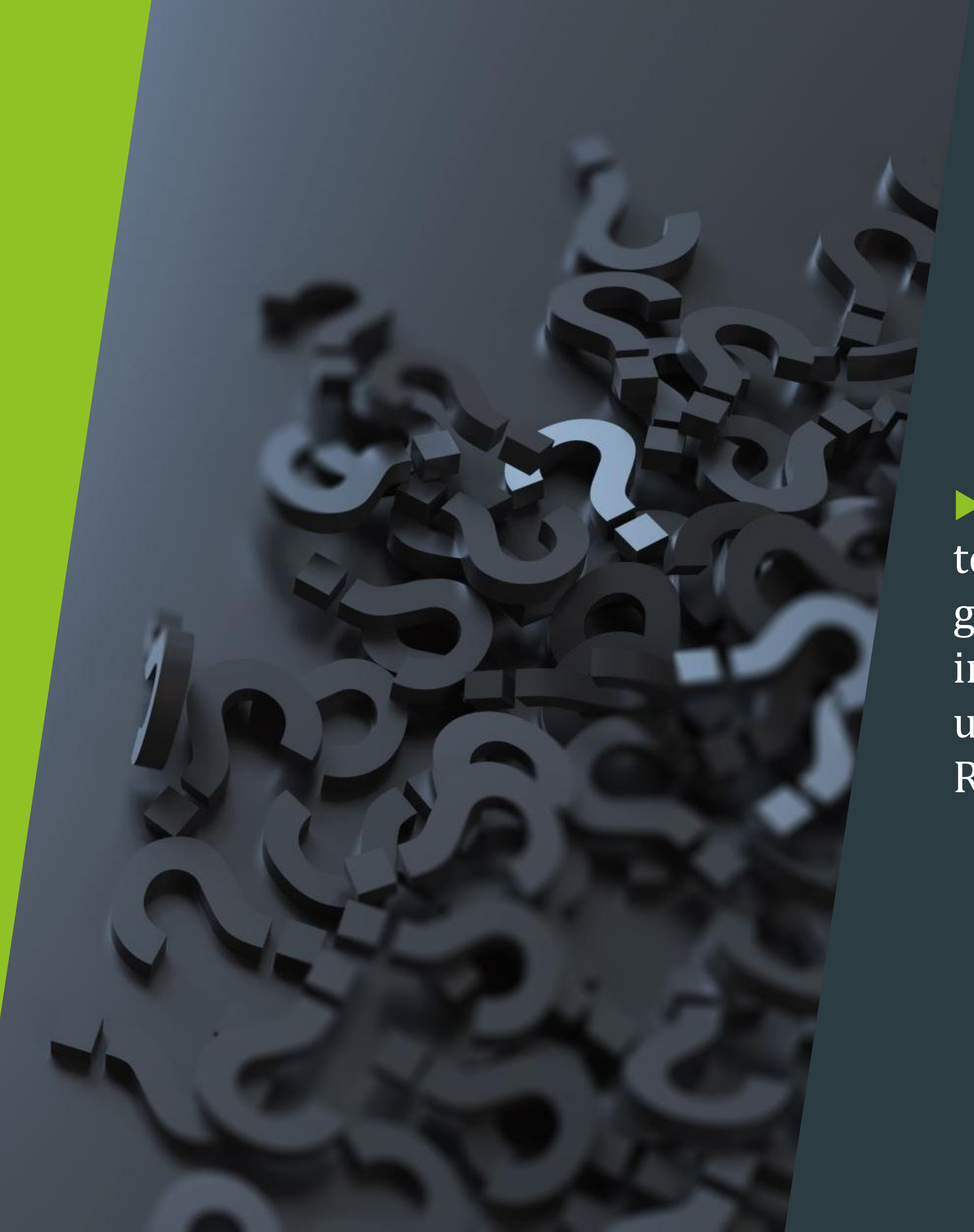
$$|\Theta| := C \left| \frac{|\tilde{\eta}|_{L^2}^{\chi(X_{n+2})/12}}{\left|\widetilde{\eta'}\right|_{L^2}} \right|$$

and

$$\psi \mapsto \tau(\psi)$$

is the mirror map.

Statement for logarithmic derivative follows.



Main theorem (CDG): The mirror conjecture holds for hypersurfaces in projective space

- In other words, the “analytic torsion” contains information about genus one Gromov-Witten invariants on the mirror, and can be understood through “arithmetic Riemann-Roch”.

Thank you for your attention

CM Calabi-Yaus

- Gross-Deligne: If the Calabi-Yau varieties are defined over $\overline{\mathbb{Q}}$ and have CM, it is conjectured that periods should be expressible in terms of Γ-values.
- We can infer from arithmetic Riemann-Roch that τ_{BCOV} should have similar properties, analogoues to classical Chowla-Selberg formulas.
- We prove it for the fiber Z_0 in the mirror family.