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Some results and problems on the genericity of genuine representations.

(1) covering group

(2) Whittaker space

(3) some results

(4) some problems

Notation: F : local field of char$(F)=0$, assume $\mu_n \subseteq F^\times$
p-adic

G : split conn red group $/F$
$= G(F)$

$Y = \mathrm{Hom}(\mathbb{G}_m, \underset{\text{max split torus} \subseteq G}{T})$: co-char lattice of G

(1) covering group à la Brylinski - Deligne

Start with

$Q : Y \longrightarrow \mathbb{Z}$ quadratic form, $\underset{N(T)/T}{W}$-invariant Weyl group

$$\Downarrow$$

$$\begin{tikzcd}
\mu_n \arrow[r, hook] \arrow[d, equal] & \overline{G}_Q \arrow[r, two heads] & G \\
\mu_n \arrow[r, hook] & \overline{T} \arrow[r, two heads, "i"] \arrow[u, hook] & T = Y \otimes F^\times \arrow[u, hook] \\
 & Z(\overline{T}) \arrow[u, hook] \arrow[r, two heads] & T^\#
\end{tikzcd}$$

$\overline{G} = \overline{G}_Q$, central extn, i.e. $\mu_n \subseteq Z(\overline{G})$

Fact: $G = \langle T \cup \{\underset{\text{splits canonically in } \overline{G}}{U_\alpha} : \alpha \text{ roots}\} \rangle$

$\overline{G} = \langle \overline{T} \cup \{U_\alpha : \alpha \text{ roots}\} \rangle$

Here $\overline{T}$ is a nilpotent group of class 2, have commutator

$$[-,-] : \overline{T} \times \overline{T} \longrightarrow \mu_n$$

$$[y(a), z(b)] = \underbrace{(a, b)_n}_{\text{Hilbert symbol}}^{B_Q(y,z)} \qquad B_Q(y,z) \overset{\text{def}}{=} Q(y+z) - Q(y) - Q(z)$$

$y, z \in Y$

$a, b \in F^\times$

$[y(a), z(b)] = (\ ,\ )_n \qquad B_Q(y,z) = Q(y+z) - Q(y) - Q(z)$

$y, z \in Y$ $(\ ,\ )_n$: n-th Hilbert symbol

$a, b \in F^\times$

Regarding $Z(\overline{T})$:

Take $Y_{Q,n} = \{ y \in Y : B_Q(y, z) \in n\mathbb{Z} \} \subseteq Y$

Then get $\quad i : Y_{Q,n} \otimes F^\times \to \underbrace{Y \otimes F^\times}_{T}$

$Z(\overline{T}) = q^{-1}(\mathrm{Im}(i))$

upshot: $\quad Y \quad$ "controls" $\quad \overline{T}$

$\qquad\quad Y_{Q,n} \qquad\qquad Z(\overline{T})$

A repn (π, V_π) of $\overline{G}$ is called genuine if μ_n acts on V_π via a fixed embedding $\mu_n \hookrightarrow \mathbb{C}^\times$.

Get $\mathrm{Irr}_{gen}(\overline{G})$.

Fact: For every $\pi \in \mathrm{Irr}_{gen}(\overline{T})$, $\dim(\pi) = \sqrt{[\overline{T} : Z(\overline{T})]}$

(Stone-von Neumann theory for Heisenberg group)

3 steps:

① pick $\underset{\text{genuine}}{\chi} : Z(\overline{T}) \to \mathbb{C}^\times$

② extend to $\chi' : \underset{\substack{\text{maximal} \\ \text{abelian subgp} \\ \subseteq \overline{T}}}{A} \to \mathbb{C}^\times$

③ induction $i(\chi) = \mathrm{Ind}_A^{\overline{T}} \chi' \in \mathrm{Irr}_{gen}(\overline{T})$

Get genuine principal series $I(\chi) := \mathrm{Ind}_{\overline{B}}^{\overline{G}} i(\chi)$.

$\overline{B} = \overline{T} \cdot U$ Borel subgp of $\overline{G}$

(2) Whittaker space.

Look at $\quad \begin{matrix} \overline{G} \\ \cup \\ U \end{matrix} \xrightarrow{\psi} \mathbb{C}^\times$, nondegenerate character,

i.e. $\psi|_{\underset{\text{simple}}{U_\alpha}} \neq \mathbb{1}$

For any $\pi \in \mathrm{Irr}_{gen}(\overline{G})$, define

$$\mathrm{Wh}_\psi(\pi) := \mathrm{Hom}_{\overline{G}}(\mathrm{ind}_U^{\overline{G}} \psi, \pi)$$

<u>Problem</u>:

describe the group homo

$$\dim_{\mathbb{C}} \mathrm{Wh}_\psi(-) : \underbrace{\mathcal{R}(\mathrm{Irr}_{gen}(\overline{G}))}_{\substack{\text{Grothendieck group} \\ \text{of } \mathrm{Irr}_{gen}(\overline{G})}} \longrightarrow \mathbb{Z}$$

In particular, fiber over $0, 1 \in \mathbb{Z}$?

Call $\pi \in \mathrm{Irr}_{gen}(\overline{G})$ generic if $\dim \mathrm{Wh}_\psi(\pi) \geq 1$

(3) Some results (selective)

(1) linear G: (Gelfand-Kazhdan 1971, Rodier 1972, Shalika 1974)

$$\dim \mathrm{Wh}_\psi(\pi) \leq 1 \text{ for every } \pi \in \mathrm{Irr}(G)$$

(Rodier 1975, Moeglin-Waldspurger 1987)

$$\dim \mathrm{Wh}_\psi(\pi) = c_{\mathcal{O}_{reg}}(\pi)$$

(2) covering $\overline{G}$:

* $\dim \mathrm{Wh}_\psi(I(\chi)) \underset{\substack{\text{Rodier} \\ \text{heredity}}}{=} \dim i(\chi) = \sqrt{[\overline{T} : Z(\overline{T})]}$
* $\dim \mathrm{Wh}_\psi(\pi) < \infty$. Kazhdan-Patterson (84), Patel (2015)
* theta repn K-P (84), G. (2017)
* depth-zero supercusp. Blondel (1992) for $\overline{GL_r}$
  G.-Weissman (2019) for $\overline{G}$.
* Have: $\dim \mathrm{Wh}_\psi(\pi) \leq 1$ for all $\pi \in \mathrm{Irr}_{gen}(\overline{G})$

$$\Updownarrow$$

$$Z(\overline{T}) = \overline{T}$$

(G.-Shahidi-Szpruch 2017)

Look at $I(\chi)$. Assume $I(\chi)$ is unramified

Then $\dim \mathrm{Wh}_\psi(I(\chi)) = |\underbrace{\mathcal{X}_{Q,n}}_{\substack{\| \\ Y/Y_{Q,n}}}|$

Question: Consider

$$I(\chi)^{s.s} = \bigoplus m_i \cdot \pi_i$$

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$$I(\chi)^{s.s} = \bigoplus_{i\in I} \underbrace{m_i}_{\geq 1} \cdot \underbrace{\pi_i}_{\in \mathrm{Irr}_{gen}(\overline{G})}$$

Then what is $\dim \mathrm{Wh}_\psi(\pi_i)$, $i \in I$?

(Note $\displaystyle\sum_{i\in I} m_i \dim \mathrm{Wh}_\psi(\pi_i) = |\mathcal{X}_{Q,n}|$)

Answer (expected, in crude form)

$$\dim \mathrm{Wh}_\psi(\pi_i) = \langle \underbrace{\sigma(\pi_i)}_{\in \mathrm{Rep}(W)}, \underbrace{\sigma_{\mathcal{X}_{Q,n}}}_{\in \mathrm{Rep}(W)} \rangle_W := \dim \mathrm{Hom}_W(\sigma(\pi_i), \sigma_{\mathcal{X}_{Q,n}})$$

The 'universal $\sigma_{\mathcal{X}_{Q,n}}$' is given as follows:

$$\underset{\substack{\text{usual action}\\ w(y)}}{W \curvearrowright Y} \Rightarrow \underset{\substack{\text{twisted action}\\ w[y] = w(y+\rho)-\rho \\ \rho = \frac{1}{2}\sum_{\alpha>0} \alpha^\vee}}{W \curvearrowright Y} \Rightarrow \underset{w[\cdot]\text{ well-defined}}{W \curvearrowright Y/Y_{Q,n}} =: \mathcal{X}_{Q,n}$$

This gives us a permutation repn

$$\sigma_{\mathcal{X}_{Q,n}} : W \longrightarrow \mathrm{Perm}(\mathcal{X}_{Q,n})$$

given by $w[\cdot]$

What is $\sigma(\pi_i)$?

Look at two cases: Case I: χ is regular

case II: χ is unitary

Case I: χ regular unramified.

Consider $\Phi(\chi) = \{ \alpha \text{ a root} : \chi(\underbrace{\overline{\alpha^\vee(a)}^{\,n_\alpha}}_{\in Z(\overline{T})}) = |a|_F \}$

$$n_\alpha = \frac{n}{\gcd(n, Q(\alpha^\vee))}$$

Then (Rodier 1981)

(1) $I(\chi)^{s.s}$ is multiplicity-free

(2) There is a natural bijection

$$\mathcal{P}(\Phi(\chi)) \longleftrightarrow JH(I(\chi))$$

$$S \longmapsto \pi_S$$

s.t. $$(\pi_S)_U = \bigoplus_{w \in W_S} \delta_B^{1/2} \cdot i({}^{w^{-1}}\chi)$$

$$W_S = \{ w \in W : \Phi(\chi) \cap w(\bar{\Phi}) = S \}$$

(3) $\pi_{\Phi(\chi)}$ is the unique unramified piece of $I(\chi)$

Thm (G. 2020)

Assume: (1) $\bar{G}$ is a persistent cover (e.g. if $\bar{G}^\vee$ is of adjoint type)

(2) $I(\chi)$ regular unramified with

$$\Phi(\chi) \subseteq \{\text{simple roots}\}$$

Then for every $S \subseteq \Phi(\chi)$, one has

$$\dim Wh_\psi(\pi_S) = \langle \underset{\sim}{\sigma(\pi_S)}, \sigma_{\mathscr{X}} \rangle_W$$

In this case, $W_S \subseteq W$ is a right cell in the sense of Kazhdan-Lusztig

$\Downarrow$

a repn $\sigma(\pi_S)$ of W

Eg $\overline{Sp}_4^{(n)}$ n odd.

$$\alpha_1 \circ\!\Leftarrow\!\circ\, \alpha_2$$

Assume $\Phi(\chi) = \{\alpha_1, \alpha_2\}$

$$S \in \{ \emptyset, \Phi(\chi), \{\alpha_1\}, \{\alpha_2\} \}$$

analogue of Steinberg $\leftarrow$ $\qquad\qquad$ $\rightarrow$ theta repn

	$\pi_\emptyset$	$\pi_{\{\alpha_1\}}$	$\pi_{\{\alpha_2\}}$	$\pi_{\Phi(\chi)}$
$\dim Wh_\psi(-)$	$\frac{n^2+4n+3}{8}$	$\frac{3(n^2-1)}{8}$	$\frac{3(n^2-1)}{8}$	$\frac{n^2-4n+3}{8}$

Case II. χ is unitary unramified.

Consider $W_\chi = \{ w \in W : {}^w\chi = \chi \} \subseteq W$

$$\Psi_\chi = \{ \alpha > 0 : \chi(\alpha^\vee(\varpi^{n_\alpha})) = 1 \}$$

$$R_\chi = W_\chi \cap W(\Psi_\chi)$$

Thm (Knapp-Stein, Silberger, W.-W. Li, C.-H. Luo)

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One has R_χ is abelian and

$$I(\chi) = \bigoplus_{\sigma \in \mathrm{Irr}(R_\chi)} \pi_\sigma$$

(normalised s.t. $\pi_{\mathbb{1}}$ is the unramified piece of $I(\chi)$)

Conj (G. 2019)

Assume: (1) G is semisimple s.c.

(2) $\overline{G}^\vee$ is of adjoint type

Then

$$\dim \mathrm{Wh}_\psi(\pi_\sigma) = \langle \underbrace{\mathrm{Ind}_{R_\chi}^{W} \sigma}_{\text{is the } \sigma(\pi_\sigma) \text{ sought for}},\ \sigma_{\mathscr{X}_{Q,n}} \rangle_W \quad \text{for every } \sigma \in \mathrm{Irr}(R_\chi).$$

All possible nontrivial R_χ:

	A_r	B_r	C_r	D_r, r even	D_r, r odd	E_6	E_7	E_8, F_4, G_2
	$\mathbb{Z}/d$, $d \mid (r+1)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$, $\mathbb{Z}/4$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	1

Conj known: if $k \cdot R_\chi = \{1\}$ for $k = 2$ or 3

difficult: A_r case

Eg $\overline{Sp}_{2r}^{(n)}$ n odd, $Q(\alpha_r^\vee) = 1$

Take χ be s.t. $\chi\left(\overline{\alpha_r(\varpi)}^{\,n}\right) = -1$

Then $R_\chi = \{1, w_{\alpha_r}\}$

$I(\chi) = \pi_{\mathbb{1}} \oplus \pi_\varepsilon$, ε = sgn char of R_χ

$\mathscr{X}_{Q,n} = (\mathbb{Z}/n)^r$

$$\dim \mathrm{Wh}_\psi(\pi_{\mathbb{1}}) = \frac{n^r + n^{r-1}}{2}, \quad \dim \mathrm{Wh}_\psi(\pi_\varepsilon) = \frac{n^r - n^{r-1}}{2}$$

$\dim \mathrm{Wh}_\psi(\pi_{\mathbb{1}})$ = # of R_χ-orbits in $\mathscr{X}_{Q,n}$, $\dim \mathrm{Wh}_\psi(\pi_\varepsilon)$ = # of free R_χ-orbits in $\mathscr{X}_{Q,n}$

(4) A problem.

Speculation/conjecture. Assume: (1) G is semisimple s.c.

(2) $\overline{G}^\vee$ is of adjoint type.

Speculation/conjecture. Assume: (1) G is semisimple s.c.

(2) $\widehat{G}^\vee$ is of adjoint type.

Then for any unramified χ, there is a character

$$\zeta_\chi^{\mathrm{ree}} : W \longrightarrow \mathbb{C}^\times \quad \text{s.t. for every } \pi \in JH(I(\chi)).$$

one has

$$\dim \mathrm{Wh}_\psi(\pi) = \Big\langle \underbrace{(\pi^I)}_{\substack{\text{Iwahori-fixed} \\ \text{vectors, a repn} \\ \text{of the Iwahori-Hecke} \\ \text{alg} \supseteq \mathbb{C}_q[W]}}{}_{q\to 1} \, , \; \zeta_\chi^{\mathrm{ree}} \otimes \sigma_{\cancel{\mathrm{Hom}}} \Big\rangle_W$$