Principal series representations of Iwahori-Hecke algebras of Kac-Moody groups over local fields

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Representations and Iwahori-Hecke algebras

- G a split reductive group over a (non-Archimedean) local field $\mathcal{F}$, (e.g $G = \mathrm{SL}_2(\mathcal{F})$),
- $\mathscr{I}$ its Iwahori subgroup (e.g $\mathscr{I} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{m} & \mathcal{O} \end{pmatrix} \cap \mathrm{SL}_2(\mathcal{F})$),
- $\mathcal{H}$ its Iwahori-Hecke algebra:

$$\begin{aligned} \mathcal{H} &= \{f : G \to \mathbb{C},\ \mathscr{I} \text{ bi-invariant, with compact support}\} \\ &= \{f : \mathscr{I}\backslash G/\mathscr{I} \to \mathbb{C}, \text{with finite support}\}, \end{aligned}$$

- V smooth representation of G, then $V^{\mathscr{I}}$ is a representation of $\mathcal{H}$,
- $V \mapsto V^{\mathscr{I}}$ induces a bijection between $\{$irreducible representations V of G s.t $V^{\mathscr{I}} \neq 0\}$ and $\{$irreducible representations of $\mathcal{H}\}$.

Principal series representations

- $T \subset G$ maximal split torus, Y cocharacter lattice of G, B Borel subgroup of G (e.g $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$),
- $T_{\mathbb{C}} = \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathbb{C}^*) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[Y], \mathbb{C}) \setminus \{0\}$,
- $\tau \in T_{\mathbb{C}} \rightsquigarrow \tau : B \to \mathbb{C}^*$, $I(\tau) = \mathrm{Ind}_B^G(\tau)$: principal series representation of G,
- $I_\tau = I(\tau)^{\mathscr{I}}$: principal series representation of $\mathcal{H}$,
- every irreducible representation of $\mathcal{H}$ is a quotient of some I_τ and embeds in some $I_{\tau'}$, for $\tau, \tau' \in T_{\mathbb{C}}$,
- Matsumoto and Kato gave irreducibility criteria for I_τ ('77 and '82).

Kac-Moody groups

- Kac-Moody groups (à la Tits): infinite dimensional (if not reductive) generalizations of reductive groups,
- "integrate" Kac-Moody Lie algebras,
- introduced in the 1970's,
- example: if $\mathbb{G}_0$ is a reductive group defined over $\mathcal{F}$, then $\widetilde{\mathbb{G}}_0(\mathcal{F})$:=(double central extension of)$\mathbb{G}_0(\mathcal{F}[u, u^{-1}])$ is an **affine** Kac-Moody group over $\mathcal{F}$,
- many other groups (**indefinite**) defined by generators and relations.

Kac-Moody groups over local fields

- Garland ('95): study of affine Kac-Moody groups when $\mathcal{F}$ is a non-Archimedean local field,
- Gaussent, Rousseau (2008): G acts on a "masure" (a.k.a hovel) $\mathcal{I}$: generalize Bruhat-Tits construction.
- Braverman, Kazhdan (2008, affine case), Gaussent, Rousseau (2012, general case): definition of the spherical Hecke algebra of G,
- Braverman, Kazhdan, Patnaik (2014, affine case), Bardy-Panse, Gaussent, Rousseau (2014, general case): definition of the Iwahori-Hecke algebra $\mathcal{H}$ of G,
- Bardy-Panse, Gaussent and Rousseau used masures.

Towards representations of G

- G is infinite dimensional $\rightsquigarrow$ topological issues on G,
- there exists no topological group structure on G for which $\mathscr{I}$ is open and compact (Abdellatif, H. 2017),
- what does smooth mean for a representation of G ?
- We can already study the representations of $\mathcal{H}$.

Kac-Moody datum

- $A=(a_{i,j})_{i,j\in I}\in\mathcal{M}_I(\mathbb{Z})$ be a **Kac-Moody matrix**:
  I: finite set, $a_{i,i}=2, a_{i,j}\leq 0$ and $a_{i,j}=0$ iff $a_{j,i}=0$ for $i\neq j\in I$,
- generalizes Cartan matrices.
- X, Y dual $\mathbb{Z}$-lattices, $X\leftrightarrow$ **characters**, $Y\leftrightarrow$ **cocharacters**,
- $(\alpha_i)_{i\in I}\in X^I$, $(\alpha_i^\vee)\in Y^I$ free families s.t $\alpha_i(\alpha_j^\vee)=a_{j,i}$ for $i,j\in I$: **simple roots, coroots**,
- $\mathbb{A}=Y\otimes\mathbb{R}$: "**standard apartment**".

Weyl group and (real) root systems

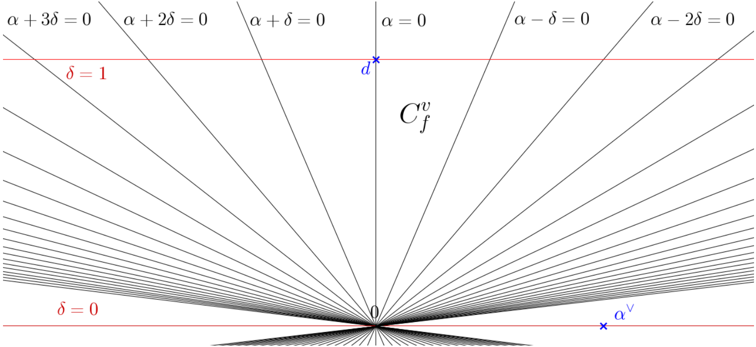
- $i \in I$: $r_i : \begin{array}{c} \mathbb{A} \to \mathbb{A} \\ x \mapsto x - \alpha_i(x)\alpha_i^\vee \end{array}$: **simple reflection**,
- $W = \langle r_i, i \in I \rangle$: **Weyl group**, Coxeter group,
- $\Phi = W.\{\alpha_i | i \in I\}$, $\Phi^\vee = W.\{\alpha_i^\vee | i \in I\}$: **root and coroot systems**,
- Φ, $\Phi^\vee$ and W are infinite (unless A is Cartan),

Kac-Moody groups

- $\mathbb{G}$: split Kac-Moody group associated with (A, X, Y), $\mathbb{G} : \{\text{fields}\} \to \{\text{groups}\}$.
- $G = \mathbb{G}(\mathcal{F})$ ($\mathcal{F}$: non-Archimedean local field),
- example: $A = (2)$, $G = \mathrm{SL}_2(\mathcal{F})$, $\mathbb{A} = \mathbb{R}$, $\alpha = \mathrm{Id}_{\mathbb{R}}$, $\Phi = \{-\alpha, \alpha\}$,

Affine SL_2: $\widetilde{\mathrm{SL}_2}$

- G =(double central extension of)$\mathrm{SL}_2(\mathcal{F}[u,u^{-1}])$, $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.
- $\mathbb{A} = \mathbb{R}\alpha^\vee \oplus \mathbb{R}c \oplus \mathbb{R}d$, $\alpha^\vee, c, d$: symbols,
- $\alpha : \mathbb{A} \to \mathbb{R}$, $\alpha(\alpha^\vee) = 2$, $\alpha(c) = \alpha(d) = 0$, $\delta : \mathbb{A} \to \mathbb{R}$, $\delta(d) = 1$, $\delta(c) = \delta(\alpha^\vee) = 0$,
- $\Phi = \{\pm\alpha + k\delta | k \in \mathbb{Z}\}$, $W = D_\infty$: infinite dihedral group.



- $\mathscr{I}$: Iwahori subgroup of G, fixator of some chamber in the masure $\mathcal{I}$ of G,
- if $G = \widetilde{\mathrm{SL}}_2 = \mathrm{SL}_2(\mathcal{F}[u, u^{-1}])$, then $\mathscr{I} = \begin{pmatrix} \mathcal{O}[u] + \mathfrak{m}[u, u^{-1}] & \mathcal{O}[u] + \mathfrak{m}[u, u^{-1}] \\ u\mathcal{O}[u] + \mathfrak{m}[u, u^{-1}] & \mathcal{O}[u] + \mathfrak{m}[u, u^{-1}] \end{pmatrix} \cap G$,
- G^+: sub-semi-group of G, $G^+ \subsetneq G$, unless G reductive,
- **Iwahori-Hecke algebra of G:** $\mathcal{H}^+ = \{\phi : \mathscr{I} \backslash G^+ / \mathscr{I} \to \mathbb{C} | \ \mathrm{supp}(\phi) \text{ is finite}\}$,
- if $\phi, \phi' \in \mathcal{H}^+$, $h \in G^+$, $\phi * \phi'(h) = \sum_{g \in G^+/\mathscr{I}} \phi(g)\phi'(g^{-1}h)$.

Bernstein-Lusztig presentation of $\mathcal{H}^+$

- $\mathcal{H}^+$ embeds in the **Bernstein-Luzstig Hecke algebra** $\mathcal{H} = \bigoplus_{\lambda \in Y, w \in W} \mathbb{C} Z^\lambda * T_w$, where Z^λ, T_w are symbols, $q = |\mathcal{O}/\mathfrak{m}|$ and for $\lambda, \mu \in Y$, $w \in W$ and $i \in I$:

1. $T_i := T_{r_i}$, $T_i * T_w = \begin{cases} T_{r_i w} \text{ if} & r_i w > w \\ (q-1) T_w + q T_{r_i w} \text{ if} & r_i w < w \end{cases}$
2. $Z^\lambda * Z^\mu = Z^{\lambda+\mu}$,
3. $Z^\lambda * T_i = T_i * Z^{r_i . \lambda} + (q-1) \frac{Z^\lambda - Z^{r_i . \lambda}}{1 - Z^{-\alpha_i^\vee}}$.

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Principal series representations of $\mathcal{H}$

- $\tau \in T_{\mathbb{C}} = \mathrm{Hom}(Y, \mathbb{C}^*) = \mathrm{Hom}(\mathbb{C}[Y], \mathbb{C}) \setminus \{0\}$,
- $I_\tau = \mathrm{Ind}_{\mathbb{C}[Y]}^{\mathcal{H}}(\tau)$,
- $I_\tau = \mathcal{H}.\mathrm{v}_\tau = \bigoplus_{w \in W} \mathbb{C} T_w.\mathrm{v}_\tau$, v_τ: symbol.

Irreducibility criteria

Weights and intertwining operators

- M an $\mathcal{H}$-module, $\tau' \in T_{\mathbb{C}}$,
- $M(\tau') = \{x \in M |\ \theta.x = \tau'(\theta).x\ \forall \theta \in \mathbb{C}[Y]\}$,
- $\mathrm{Wt}(M) = \{\tau' \in T_{\mathbb{C}} | M(\tau') \neq \{0\}\}$,

Irreducibility criterion

- Theorem (H. '19):
  - Let $\tau \in T_{\mathbb{C}}$. Then I_τ is irreducible iff:
    1. $\tau(\alpha^\vee) \neq q$ for every $\alpha^\vee \in \Phi^\vee$,
    2. $\mathrm{End}_{\mathcal{H}}(I_\tau) = \mathbb{C}\mathrm{Id}$ (or equivalently $I_\tau(\tau) = \mathbb{C}v_\tau$).

Irreducible quotients of regular principal series representations

- **Theorem:** (Rodier, Rogawski '80: G reductive, H'20: Kac-Moody case):

  Let $\tau \in T_{\mathbb{C}}$ be regular ($W_\tau = \{1\}$). Then:

(1) If M is a submodule or a quotient of I_τ, then $M = \bigoplus_{\tau' \in \mathrm{Wt}(M)} M(\tau')$, $\dim M(\tau') = 1$ for every $\tau' \in \mathrm{Wt}(M)$ and $\mathrm{Wt}(M) \subset W.\tau$,

(2) If $w \in W$, there exists a unique irreducible $\mathcal{H}$-module $M_{w.\tau}$ such that $M_{w.\tau}(w.\tau) \neq \{0\}$. Then $\dim M_{w.\tau} = |\mathrm{Wt}(M)| = |\{w' \in W^v | I_{w.\tau} \simeq I_{w'.\tau}\}|$.

Irreducible quotients of regular principal series representations

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- example: $G = \mathrm{SL}_3(\mathcal{F})$, $W = \langle s, t\rangle$, $\tau(\alpha_s^\vee) = \tau(\alpha_t^\vee) = q^{\pm 1}$,

$I_{s.\tau} \simeq I_{ts.\tau}$

$I_\tau \not\simeq I_{s.\tau}$, $\quad I_{ts.\tau} \not\simeq I_{sts.\tau}$

$I_{sts.\tau} = I_{tst.\tau}$

$I_\tau \not\simeq I_{t.\tau}$, $\quad I_{st.\tau} \not\simeq I_{tst.\tau}$

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Irreducible quotients of regular principal series representations

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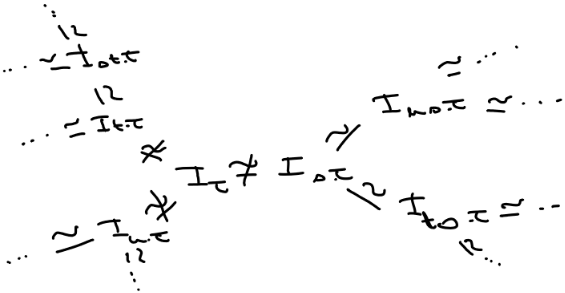
- example: $A = \begin{pmatrix} 2 & * & * \\ * & 2 & * \\ * & * & 2 \end{pmatrix}$, with $* \leq -2$, $W = \langle s, t, u \rangle = \langle s, t, u | s^2 = t^2 = u^2 = 1 \rangle$, $\tau(\alpha_s^\vee) = \tau(\alpha_t^\vee) = \tau(\alpha_u^\vee) = q^{\pm 1}$,



Irreducible quotients of regular principal series representations

(2) If $w \in W$, there exists a unique irreducible $\mathcal{H}$-module such that $M_{w.\tau}(w.\tau) \neq \{0\}$. Then
$\dim M_{w.\tau} = |\mathrm{Wt}(M)| = |\{w' \in W | I_{w'.\tau} \simeq I_{w.\tau}\}|$,

- example: $G = \widetilde{\mathrm{SL}}_2(\mathcal{F})$, $W = \langle s, t \rangle \simeq D_\infty$, there exists $\tau \in T_\mathbb{C}$ regular such that:

$$\cdots \not\simeq I_{st.\tau} \not\simeq I_{t.\tau} \not\simeq I_{\tau} \not\simeq I_{s.\tau} \not\simeq I_{ts.\tau} \not\simeq \cdots$$

Case where $\tau(\alpha^\vee) \neq q$ for all $\alpha^\vee \in \Phi^\vee$

- Let $\tau \in T_{\mathbb{C}}$ such that $\tau(\alpha^\vee) \neq q$ for all $\alpha^\vee \in \Phi^\vee$. Then $I_\tau \simeq I_{w.\tau}$, for every $w \in W$,
- If $J \subset \mathrm{End}_{\mathcal{H}-\mathrm{mod}}(I_\tau)$, $J(I_\tau) = \sum_{\phi \in J} \phi(I_\tau)$.
- Theorem (H. '20, A size 2 Kac-Moody matrix, not Cartan)

  $J \mapsto J(I_\tau)$ is a bijection between $\{$right ideals of $\mathrm{End}(I_\tau)\}$ and $\{$submodules of $I_\tau\}$,
- $\mathrm{End}(I_\tau) \simeq \mathbb{C}[R_\tau]$, where $R_\tau = W_\tau / W_{(\tau)}$,
- $R_\tau \in \{\{1\}, \langle r \rangle \simeq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}, \langle r, s \rangle \simeq D_\infty\}$
- $M \mapsto I_\tau / M$ is a surjection from $\{$maximal submodules of $I_\tau\}$ to $\{$irreducible $\mathcal{H} -$ modules M $s.t$ $\tau \in \mathrm{Wt}(M)\}$. It is a bijection iff every maximal ideal of $\mathrm{End}(I_\tau)$ is two-sided. In this case, these representations have dimension $|W_{(\tau)}||W^v / W_\tau|$.

Link with representations of G?

- Suppose G is reductive.
- $I(\tau)$ set of locally constant $f : G \to \mathbb{C}$ such that

$$f(bg) = \tau\delta^{1/2}(b)f(g),\ \forall g \in G, b \in B,$$

- $\delta^{1/2} : B \to \mathbb{C}^*$: modulus character, $G \curvearrowright I(\tau)$ by right translation,
-

$$\begin{aligned}\forall(\phi, f) \in \mathcal{H} \times I(\tau)^{\mathscr{I}}, \phi.f &= \int_G \phi(g)g.fd\mu(g)\\ &= \mu(\mathscr{I}) \sum_{g \in G/\mathscr{I}} \phi(g)g.f.\end{aligned}$$

Link with representations of G?

- Suppose G Kac-Moody,
- $\widehat{I(\tau)} = \{f : G \to \mathbb{C} | f(bg) = \tau\delta^{1/2}(b)f(g),\ \forall b \in B, g \in G\}$,
- no regularity assumption on f (for the moment?),
- $I_{\tau,G}$: set of functions in $\widehat{I(\tau)}^{\mathscr{I}}$ satisfying some finiteness support condition,
- Proposition (H. '21): $\mathcal{H} \curvearrowright I_{\tau,G}$:

$$\forall(\phi, f) \in \mathcal{H}_{\mathbb{C}} \times I_{\tau,G}, \phi.f := \sum_{g \in G/\mathscr{I}} \phi(g)g.f$$

is well defined (finiteness issues) and $I_{\tau,G} \simeq I_\tau$ as an $\mathcal{H}$-module.