Green forms, special cycles and modular forms

University of Alberta, GNTRT Seminar

Siddarth Sankaran (University of Manitoba)

April 13 2021

Theta functions

- Let $(V, Q) =$ quadratic space $/\mathbb{Q}$
- Assume (V, Q) *positive definite*, $\dim V = m$ even
- Fix $L \subset V$ lattice, $Q(L) \subset \mathbb{Z}$

Theta function:

$$\Theta_L(\tau) := \sum_{x \in L} e^{2\pi i Q(x)\tau} \qquad \tau \in \mathbb{C}, Im(\tau) > 0$$

Then $\Theta_L(\tau)$ is *modular*: $\exists \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ finite index s.t.

$$\Theta_L\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{m/2}\Theta_L(\tau), \qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$$

[If L self-dual, can take $\Gamma = \mathrm{SL}_2(\mathbb{Z})$]

Theta functions

$$\Theta_L(\tau) = \sum_{x \in L} e^{2\pi i Q(x)\tau} = \sum_{n \geq 0} r_L(n) q^n \qquad q = e^{2\pi i \tau}$$

where $r_L(n) = \#\{x \in L \mid Q(x) = n\}$ = representation number (arithmetic)

- Moral: modularity of $\Theta_L(\tau)$ encodes subtle symmetries/relations between $r_L(n)$.

Siegel-Weil formula

- For simplicity, consider even unimodular lattices:
  $\mathcal{C} := \{L \subset V \mid L^\vee = L, Q(L) \subset \mathbb{Z}\}/isom.$
- Define Eisenstein series of weight $k > 2$, $s \in \mathbb{C}$, $Re(s) \gg 0$

$$E_k(\tau, s) := \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \frac{Im(\gamma\tau)^{\frac{1}{2}(s-k+1)}}{(c\tau + d)^{2k}}, \qquad \Gamma_\infty = \left\{ \left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right) \right\}$$

Theorem (Siegel, 1930's)

$$\sum_{L \in \mathcal{C}} \frac{1}{\#\mathrm{Aut}(L)} \Theta_L(\tau) = C \cdot E_{m/2}(\tau, m/2 - 1),$$

where $m = \dim V$ *and* $C =$ *explicit const*

Vastly generalized by Weil using representation theoretic approach

Orthogonal symmetric spaces

- F = totally real field, embeddings $\sigma_1, \dots, \sigma_d \colon F \to \mathbb{R}$
- V = vector space over F
- $Q \colon V \to F$ quadratic form, **anisotropic**
- *Signature condition:* $V_i := V \otimes_{F,\sigma_i} \mathbb{R}$ is a quadratic space $/\mathbb{R}$.
  Assume: V_1 has signature $(m-2,2)$, V_i is pos def for $i > 1$

Symmetric space:

$$\mathbb{D} := \{ z \subset V \otimes_{F,\sigma_1} \mathbb{R} \mid \dim_{\mathbb{R}}(z) = 2,\ Q|_z \text{ negative definite} \}$$

$\mathbb{D}$ is a cpx manifold, $\dim_{\mathbb{C}}(\mathbb{D}) = m-2$

Orthogonal Shimura varieties

- Fix lattice $L \subset V$
- For $\Gamma \subset O(L)$ sufficiently small, have Shimura variety:

$$X = X_\Gamma := \Gamma \backslash \mathbb{D}$$

- X is projective, has model over some number field E

Examples:

- $V = \{A \in M_2(\mathbb{Q}) \mid tr(A) = 0\} \qquad Q(A) = N \det A$
  $L := \{\begin{pmatrix} a & b/N \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{Z}\}$.
  $\implies \mathbb{D} \simeq \mathbb{H}$, and $X = Stab(L) \backslash \mathbb{D} \simeq \Gamma_0(N) \backslash \mathbb{H}$
- Can also describe Hilbert modular surfaces, Picard surfaces, etc. in this way

Special cycles (Kudla)

$\mathbb{D} := \{z \subset V \otimes_{F,\sigma_1} \mathbb{R} \mid \dim_{\mathbb{R}}(z) = 2,\ Q|_z \text{ negative definite}\}$

- Fix $n > 0$, let $x = (x_1, \ldots, x_n) \in L^n$, $\quad \Gamma_x = \mathrm{Stab}_\Gamma(x)$

$$\mathbb{D}_x := \{z \in \mathbb{D} \mid z \perp x_i \text{ for } i = 1, \ldots, n\},$$

$\rightsquigarrow$ obtain cycle $Z(x)\colon \Gamma_x \backslash \mathbb{D}_x \to \Gamma \backslash \mathbb{D}$

For $T = (T_{ij}) \in \mathrm{Sym}_n(F)$, let $\Omega(T) := \{x \in L^n | \langle x_i, x_j \rangle = T_{ij}\}$

$$Z(T) := \sum_{\substack{x \in \Omega(T) \\ \text{mod } \Gamma}} Z(x)$$

- If T positive semi def ($+$ $Z(T)$ non-empty) then $codim Z(T) = rk(T)$
- $Z(T)$ is algebraic, defined over E

Generating series

Theorem (Kudla-Millson, special case)

$$\Theta_{KM}(\tau) := \sum_T \{Z(T)\} q^T, \qquad \{Z(T)\} \in H^{2n}_{dR}(X)$$

is a Siegel modular form (= automorphic form for $Sp_{2n}(F)$) of parallel scalar weight $m/2$.

Idea of proof: for $x \in V^n$, KM construct a closed diff form $\varphi_{KM}(x)$ that behaves like a Gaussian fn (i.e. exp decay in x), so can be used to form a theta fn $\tilde{\Theta}(\tau)$ valued in closed $2n$ forms (a priori modular).

Moreover $\{\sum_{x \in \Omega(T)} \varphi_{KM}(x)\} = \{Z(T)\}$ so $\Theta_{KM}(\tau) = \{\tilde{\Theta}(\tau)\}$.

Generating series

Consider

$$\Theta_{CH}(\tau) := \sum_{T} [Z(T)] q^{T}, \qquad [Z(T)] \in CH^{n}(X)$$

Modularity results:

- $n = 1$: Borcherds, Yuan-Zhang-Zhang ($F \neq \mathbb{Q}$)
- $F = \mathbb{Q}$, $n > 1$: W. Zhang + Bruinier-Raum
- $F \neq \mathbb{Q}$, $n > 1$: conditional proofs Kudla, Maeda

Arithmetic generating series

- Gross-Zagier: intepret as identity b/w *arithmetic* intersection # of Heegner divisors (= special divisors on $X_0(N)$) and *derivative* of L-function
- Kudla program: generalization to arithmetic models of orthogonal Shimura varieties.
  Roughly: suppose have "nice" integral model $\mathcal{X}$ over $Spec(\mathcal{O}_E)$, and "nice" classes $\widehat{\mathcal{Z}}(T)$ in arithmetic Chow gp of $\mathcal{X}$ (more on this in a moment).
  Then expect
  $$\sum_T \widehat{\mathcal{Z}}(T) q^T$$
  is modular

Arithmetic Siegel-Weil

$\mathcal{X}$ = nice model of $X = \Gamma\backslash\mathbb{D}$, $\widehat{\mathcal{Z}}(T)$ arithmetic cycle

- $\exists$ "tautological" metrized bundle $\widehat{\omega}$ on X; assume extn to $\mathcal{X}$

Arithmetic Siegel-Weil conjecture (Kudla)

$$\sum_{T \in Sym_n(O_F)} \langle \widehat{\mathcal{Z}}(T), \widehat{\omega}^{m-n+1} \rangle q^T \sim C \cdot E' \left(\tau, \frac{m-n-1}{2} \right)$$

- $\langle \cdot, \cdot \rangle$ is an intersection pairing on arith Chow gp
- $E(\tau, s)$ Siegel Eisenstein series (= Eisenstein series for Sp_{2n})

Construction of models and arithmetic cycles is very complicated in general, but lots of evidence for conjectures in particular cases: e.g. Kudla-Rapoport-Yang (Shimura curves), works of Kudla, Rapoport, Howard, Bruinier, Yang, Li, Wei Zhang, many others...

Arithmetic Chow groups (Gillet-Soulé)

- $\mathcal{X}$ = projective variety over "arithmetic ring" R
  (e.g. can take R = number field, O_E, $O_E[1/N]$, ...)
- $X = \mathcal{X}(\mathbb{C}) := \coprod_{\sigma\colon R \hookrightarrow \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$

An *arithmetic cycle* of codim n is a pair $\widehat{\mathcal{Z}} = (\mathcal{Z}, g)$:

- $\mathcal{Z}$ = codim n cycle on $\mathcal{X}$ (with $\mathbb{C}$-coeffs, say)
- $g \in D^{n-1,n-1}(X)$: degree $(n-1, n-1)$ current
- **Green's equation:** $\exists$ a smooth form ω s.t.
  $\mathrm{dd^c} g + \delta_{\mathcal{Z}(\mathbb{C})} = [\omega]$ $[\mathrm{dd^c} = \frac{1}{2\pi i}\partial\overline{\partial}]$

$$\widehat{CH}^n(\mathcal{X})_{\mathbb{C}} := \{\text{arith cycles}\}/\text{"rational equivalence"}$$

[$D^{n-1,n-1}(X)$ = cts linear functionals on $A^{n',n'}(X)$, $n' = \dim X - n + 1$]

Green currents for special cycles

Back to $X = \Gamma\backslash\mathbb{D}$, $Z(T)$, etc. To define arith cycle $\widehat{\mathcal{Z}}(T)$, need to specify (independently)

1. integral models $\mathcal{X}$, $\mathcal{Z}(T)$
2. Green current g for $Z(T) = \mathcal{Z}(T)(\mathbb{C})$

(Garcia-S.) Construction of a current $g(T, v)$ via Quillen's theory of superconnections:

- Depends on parameter $v \in Sym_n(F \otimes_{\mathbb{Q}} \mathbb{R})_{>0}$
- Functorial (e.g. $O(n)$ invariance, compatible with embeddings of Sh vars)
- $*$-product identity: if $Z(T), Z(T')$ intersect properly, then
$$g(T, v) * g(T', v') \equiv \sum_{S = \begin{pmatrix} T & * \\ * & T' \end{pmatrix}} g\left(S, \begin{pmatrix} v & \\ & v' \end{pmatrix}\right)$$
mod exact currents

Construction of $g(T, v)$ (suppose T non-degenerate)

- Garcia: $\mathbb{D}_x$ = zero locus of section of a Herm bundle on $\mathbb{D}$
  $\implies$ Quillen: "super Chern form" $= \varphi_{KM}(x) \in A^{2n}(X)$
- Trangression (Bismut-Gillet-Soulé): $\exists$ explicit form $\nu(x)$ st

$$\mathrm{dd^c}\nu(\sqrt{t}x) = -t\frac{\partial}{\partial t}\varphi(\sqrt{t}x), \qquad t \in \mathbb{R}_{>0}$$

- Define $g(x) := \int_1^\infty \nu(\sqrt{t}x)\frac{dt}{t}$ [assume $x \in V^n$ lin indep]

$$\begin{aligned}\implies \mathrm{dd^c}g(x) &= \int_1^\infty \mathrm{dd^c}\nu(\sqrt{t}x)\frac{dt}{t} = -\int_1^\infty \frac{\partial}{\partial t}\varphi(\sqrt{t}x)dt \\ &= \varphi(x) - \lim_{t\to\infty}\varphi(\sqrt{t}x) \\ &= \varphi(x) - \delta_{\mathbb{D}_x} \qquad [\textit{Bismut}]\end{aligned}$$

Define $g(T, v)$ by averaging over $x \in \Omega(T)$, descend to $X = \Gamma\backslash\mathbb{D}$

Arithmetic heights

Assume good int models $\mathcal{X}$, $\mathcal{Z}(T)$
$\rightsquigarrow \widehat{\mathcal{Z}}(T,v) := (\mathcal{Z}(T), g(T,v)) \in \widehat{CH}^n(\mathcal{X})$

Kudla's conjecture: $\langle \widehat{\mathcal{Z}}(T,v), \widehat{\omega}^{m-n+1} \rangle \sim C \cdot E_T'(v, s_0)$

Decomposition of arithmetic intersection pairing:

$$\langle \widehat{\mathcal{Z}}(T,v), \widehat{\omega}^{m-n+1} \rangle = \int_X g(T,v) \wedge \Omega^{m-n+1} + ht_{\widehat{\omega}}(\mathcal{Z}(T))$$

[Ω = Kähler form on X]

Theorem (Garcia-S.)

$\exists$ *explicit constant* $\kappa(T)$ *such that*

$$\int_X g(T,v) \wedge \Omega^{m-n+1} = C \cdot E_T'(v, s_0) + \kappa(T)$$

Non-holomorphic parts (depending on v) in conjecture match

Another application

Take $\mathcal{X} = X$, defined over E.

Let $\widehat{Z}(T, v) = (Z(T), g(T, v)) \in \widehat{CH}^n_{\mathbb{C}}(X, D_{cur})$

[arith Chow group due to Burgos-Kramer-Kühn]

Fix $T_2 \in \mathrm{Sym}_{n-1}(F)$, and consider partial gen series

$$\phi_{T_2}(\tau) = \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{Z}(T, v) q^T$$

Theorem (S.)

$\phi_{T_2}(\tau)$ *is a Jacobi form of weight* $m/2 + 1$*, and index* T_2*.*

Proof uses $*$-product formula for $g(T, v)$ & induction argument

Rmk: if $F(\tau) = \sum_T c(T, v) q^T$ is a Siegel mod form, then $F_{T_2}(\tau) = \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} c(T, v) q^T$ is a Jacobi mod form, so Thm gives evidence full gen series is modular.

Thank you.