Review of geometric scattering theory

Geometry of Locally Symmetric Spaces

Scattering on quotient of $SL(3, \mathbb{R})$

Scattering Theory of Locally Symmetric Spaces

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Outline

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1. Review of geometric scattering theory
2. Scattering geodesics in locally symmetric spaces of $\mathbb{Q}$-rank one
3. Scattering flats in locally symmetric spaces of higher $\mathbb{Q}$-rank
4. Main theorems and overview of proof method
5. Future Directions

Introduction

Geometric scattering theory was introduced by physicists Ludvig Faddeev and Victor Popov to establish a rigorous framework for the scattering processes that show up in Quantum field Theory. Scattering theory compares the asymptotic behavior of an evolving system as t tends to $-\infty$ with its asymptotic behavior as t tends to ∞. It is especially fruitful for studying systems constructed from a simpler system by the imposition of a disturbance (also called perturbation or scatterer) provided that the influence of the disturbance on motions at large $|t|$ is negligible.

There are several frameworks for studying geometric scattering, the one that is of particular interest to Number theory and Spectral geometry is the framework developed by Lax and Phillips.

Scattering through a compact obstacle

Let B be a compact convex domain in $\mathbb{R}^n$ and $f, g \in C_0^\infty(\mathbb{R}^n - B)$. Then one can introduce a norm on the linear space consisting of such pairs (f, g) given by $||(f, g)||$, such that $||(f, g)||^2 = \int_{\mathbb{R}^n}(|\nabla f|^2 + |g|^2)dx$. Define the Euclidean Laplacian $\Delta = \sum_{i=1}^n \partial_i^2$, where $\partial_i = \frac{\partial}{\partial x_i}$ and let $\langle , \rangle_{\mathbb{R}^n}$ be the standard norm on $L^2(\mathbb{R}^n)$. Then using integration by parts an equivalent form of this norm is obtained, namely $||(f, g)||^2 = \langle g, g \rangle_{\mathbb{R}^n} - \langle f, \Delta f \rangle_{\mathbb{R}^n}$.
Define the Hilbert space $\mathcal{H}$ to be the completion of the set of pairs (f, g) with respect to this norm.
Consider the following boundary value problem with the Euclidean Laplacian $\Delta = \sum_{i=1}^n \partial_i^2$, where $\partial_i = \frac{\partial}{\partial x_i}$.

$$v_{tt} - \Delta v = 0, \;\; v \in L^2(\mathbb{R}^n - B) \quad and \quad v = 0 \quad on \quad \partial B$$

Scattering through a compact obstacle

Note that, this has a unique solution associated with a Cauchy data of the form $v(x,0) = f$ and $v_t(x,0) = g$, where $(f,g) \in \mathcal{H}$. One can define then the time evolution of the initial data (f,g) as a linear operator $U_t^B : \mathcal{H} \longrightarrow \mathcal{H}$, given by $U_t^B(f,g) = (v, v_t)$. It can be shown that U_t^B is a unitary operator for $t \in \mathbb{R}$.

When B is the empty set, denote by $\mathcal{H}_o$ and U_o the associated Hilbert space and unitary group respectively.

Since $\mathcal{H}_o \supset \mathcal{H}_B$, the map $U_o(-t)U_B(t)$ is a well-defined map from $\mathcal{H}_B$ to $\mathcal{H}_o$.

When the dimension is odd, it is known that the limit $W^{\mp} = \lim_{t\to\mp\infty} U_o(-t)U_B(t)$ exists, and one has the associated scattering operator $S = (W^-)^{-1}W^+$.

Scattering through a compact obstacle

Next, a translational representation of the one parameter group U_o is constructed, by defining an isomorphism $\Psi : \mathcal{H}_o \longrightarrow L^2(R, M)$, where $M = L^2(S^{n-1})$ and $\Psi U_o(t)\Psi^{-1}$ is the operator which is the multiplication by $e^{it\mu}$, for some $\mu \in \mathbb{R}$. This gives rise to scattering operators $S_\mu : L^2(S^{n-1}) \longrightarrow L^2(S^{n-1})$.

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For $(\theta_1, \theta_2) \in S^{n-1} \times S^{n-1}$, let $S_\mu(\theta_1, \theta_2)$ be the Schwartz kernel of the operator S_μ. One of the major results of scattering theory relates these scattering amplitudes $S_\mu(\theta_1, \theta_2)$ with the sojourn time associated to certain scattered rays.

Scattering through a compact obstacle

For $\theta_1 \neq \theta_2 \in S^{n-1}$, the kernel $S_\mu(\theta_1, \theta_2)$ is a C^∞ function of all the three variables. Furthermore, for fixed $\theta_1 \neq \theta_2$ we have the following asymptotic expansion

$$S_\mu(\theta_1, \theta_2)(\mu/2\pi i)^{1/2-n/2} = J^{-1/2}e^{i\tau\mu} + O(1/\mu)$$

Where τ is the sojourn time of the unique scattered ray with direction of incidence at θ_1 and direction of reflection at θ_2 and J is the scattering differential cross-section at (θ_1, θ_2). Denoting by $\gamma_{\theta_1,\theta_2}$ the ray reflected off B with angle of incidence θ_1 and angle of reflection θ_2, such that this ray intersects the obstacle B for the first time at $b \in \partial B$. Then denoting by $k(b)$ the sectional curvature at b of the surface ∂B, we have

$$J = 4(2c)^{n-3}k(b)$$

Where, c is the cosine of the angle that the ray $\gamma_{\theta_1,\theta_2}$ makes with the normal to the surface ∂B at the point $b \in \partial B$.

Scattering on finite area hyperbolic surfaces

Victor Guillemin was the first one who realized that just as in the case of a compact manifold, there is an analogous poisson relation between sojourn times of scattering geodesics and singularities of the scattering matrices for non compact hyperbolic surfaces. Here we will review the work of Guillemin's paper.

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Let $\mathbb{H} = \{z = x + iy | x, y \in R, y > 0\}$ is the upper half plane with the assigned hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. The associated laplacian is given by $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$, since $\mathbb{H}$ is a complete Riemannian manifold, the laplacian has a unique self adjoint extension which is also be denoted by Δ.

Scattering on finite area hyperbolic surfaces

Now consider a cofinite discrete torsion free subgroup Γ of PSL(2,$\mathbb{R}$) and let $X = \Gamma\backslash\mathbb{H}$ be the associated finite area non-compact hyperbolic surface with $k_1, ..., k_n$ inequivalent cusps . One then knows that, for any sufficiently large "a" , X is a disjoint union of compact subset X_a and a finite number of open sets X_i , $i = 1, 2, , , , , n$, where X_i is the cusp neighbourhood for the corresponding cusp k_i and so that each X_i is isometric to the set $\{-1/2 \leq Re(z) \leq 1/2 || Im(z)| \geq a\}$ in the upper half plane.

Scattering geodesics on finite area hyperbolic surfaces

Definition

A geodesic $\gamma(t)$ in X is called a scattering geodesic if it is contained in $X \backslash X_a$ for large positive as well as negative times t. A scattering geodesics that is contained in X_i for $t \ll t_0$ and in X_j for $t_1 \ll t$ is called a geodesic scattered between cusp ends X_i to X_j . The associated **sojourn time** T_γ is the total amount of time the geodesic spends in the compact core X_0, starting from the first time it entered X_0 until the time when it exits.

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Asume the cusp k_i is at ∞ and the cusp k_j will then be a vertex of the fundamental domain lying on the real axis given by the point $(x_j, 0)$ with the cusp neighborhood bounded bytwo geodesics σ_1 and σ_2 which are perpendicular to the real axis at $(x_j, 0)$.

Scattering geodesics on finite area hyperbolic surfaces

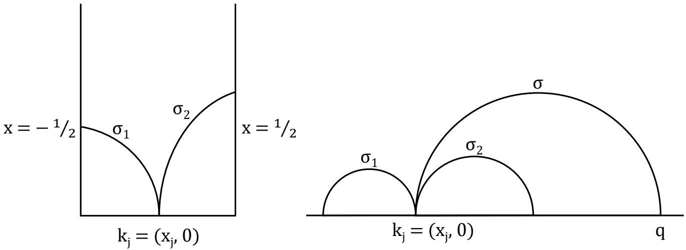

Figure: Construction of scattering geodesics.

Then for a $B \in \Gamma$ such that $Bq = \infty$, the image of $B\sigma$ in $\Gamma\backslash\mathbb{H}$ is a scattering geodesic. This shows that there are countable number of scattering geodesics running between two fixed cusp neighborhoods.

Scattering geodesics on finite area hyperbolic surfaces

We further choose an isometry Ψ mapping the vertical strip $\{-1/2 \leq Re(z) \leq 1/2\}$ onto the j-th cusp neighborhood such that $\Psi(\infty) = k_j$.
Associated to the i-th cusp, we have the **Eisenstein Series** $E_\infty(z, s)$ given by,

$$E_\infty(z, s) = \sum_{B \in \Gamma_\infty \backslash \Gamma} (Im(Bz))^s$$

Now we set $s = 1/2 + i\tau$, and let $E(z, \tau) = E_\infty(z, 1/2 + i\tau)$. Then observe that the zero-th Fourier coefficient in the expansion of $E(z, \tau)$ in the j-th cusp neighborhood is given by the integral,

$$\int_{-1/2}^{1/2} E(\Psi z, \tau) dx = e^{2i\tau \ln(a)} C_{ij}(\tau) y^{1/2 - i\tau}$$

Scattering geodesics on finite area hyperbolic surfaces

Where, $C_{ij}(\tau)$ is the *ij*-th entry of the scattering matrix.

Theorem (Guillemin)

Let $\mathcal{T}_{ij}$ be the set of sojourn times for geodesics that are scattered from the i-th cusp neighborhood to the j-th cusp neighborhood. Define the following integral,

$$F(\tau) = \int_{-\infty}^{\infty} (1 + w^2)^{-(1/2 + i\tau)} dw \qquad (1)$$

Then for $Im(\tau) \leq -3/2$, one has

$$C_{ij}(\tau) = aF(\tau) \sum_{T_\sigma \in \mathcal{T}_{ij}} e^{-T_\sigma(1/2 + i\tau)} \qquad (2)$$

For a general τ, the right-hand side is supposed to be the meromorphic continuation of this series.

Locally Symmetric spaces

Locally Symmetric spaces

Let $\mathbf{G}$ be a complex semisimple linear algebraic group defined over $\mathbb{Q}$. Denote by G the real locus $\mathbf{G}(\mathbb{R})$ of $\mathbf{G}$, then G is a real semisimple lie group with finitely many connected components. Choose a maximal compact subgroup K of G. Then the associated symmetric space $X = G\backslash K$ is a negatively curved Riemannian manifold that admits a G-action and a G-invariant Riemannian metric.

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Let $\mathfrak{g}$ be the Lie algebra of G, then $\mathfrak{g}$ admits a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K. If $B(\bullet, \bullet)$ denotes the associated Killing form of $\mathfrak{g}$, then it is known that the restriction of B to $\mathfrak{k}$ is negative definite and its restriction to $\mathfrak{p}$ is positive definite. The restriction of B to $\mathfrak{p}$ gives rise to a G-invariant Riemannian metric on X.

Locally Symmetric spaces

Choose a torsion free arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and define the Riemannian manifold $S = \Gamma \backslash X = \Gamma \backslash \ G/K$. Then, S is a locally symmetric space of finite volume. When S is non-compact(which is the case we will focus on), it has both discrete and continuous spectrum, the latter characterized by Eisenstein series and a certain set of intertwining operators called the **scattering matrices**.

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Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$, and let P be the real locus $\mathbf{P}(\mathbb{R})$. Then P has an associated rational Langlands decomposition $P = M_P A_P N_P$ where A_P is called a rational split component chosen such that it is stable under the Cartan involution associated to a fixed base point in X.

Locally Symmetric spaces

The Langlands decomposition of P gives rise to the splitting $X \simeq M_P/(K \cap P) \times A_P \times N_P$, called the horospherical decomposition associated to P . We denote by X^P the manifold $M_P/(K \cap P)$, which is the boundary symmetric space of X associated to the rational parabolic subgroup P. We also have an associated boundary locally symmetric space arising from a quotient of X^P which we will denote by S_P.

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Suppose $\mathfrak{a}_P$ is the Lie algebra of A_P, along with the diffeomorphic exponential mapping $exp_P : \mathfrak{a}_P \to A_P$. Then any $x \in X$ can be represented as $x = (z, exp_P(H), w) \in X^P \times A_P \times N_P$ and $H \in \mathfrak{a}_P$.

Spectral Theory of the Laplacian on S

Denote the natural projection map $\pi_P : X \longrightarrow X^P$ coming from the horospherical decomposition of X associated to P given by $X = X^P \times A_P \times N_P$.

Spectral Theory of the Laplacian on S

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Further let $\Sigma^+(P, A_P)$ be the set of positive roots associated to the adjoint action of $\mathfrak{a}_P = \text{Lie}(A_P)$ on $n_P = \text{Lie}N_P$. Finally let for $x \in X$, let $H_P(x)$ denote the $\mathfrak{a}_P$ component of x in the horospherical decomposition of X associated to P. Then for any eigenfunction $\phi \in L^2(S_P)$ of the Laplacian acting on S_P and $\lambda \in \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$ with $Re(\lambda) >> 0$ we have the Eisenstein Series given by

$$E_P(x, \phi, \lambda) = \sum_{\gamma \in (\Gamma \cap P)\backslash P} e^{(\tau_P + \lambda)(H_P(\gamma x))}\phi(\pi_P(\gamma x))$$

Properties of the Eisenstein Series

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Note that for $Re(\lambda) >> 0$ the series $E_P(x, \lambda, \phi)$ converges uniformly on compact subsets of X, one of the main results of Langlands is that this series admits a meromorphic continuation to the whole space $\mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$. Further note that, this series is Γ invariant by construction and descends to a smooth function on S which is smooth eigenfunction of the Laplacian on S. This Eisenstein series are the building blocks for the continuous spectrum of S.

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For a (possibly different) parabolic subgroup Q with Langlands decomposition $Q = M_Q A_Q N_Q$ define the restriction of $E_P(x, \phi, \lambda)$ along Q by the integral,

$$E_{P|Q}(x, \phi, \lambda) = \int_{(\Gamma \cap N_Q)\backslash N_Q} E_P(nx, \phi, \lambda) dn$$

Scattering Matrices

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Two rational parabolic subgroups P and Q are called associate if $\exists y \in \mathbf{G}(\mathbb{Q})$ such that $ad(y)A_P = A_Q$. There are essentially two different cases to be considered.

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If $rank(P) \geq rank(Q)$, and the two parabolic subgroups are not associate, then a result of Langlands says $E_{P|Q}(x, \phi, \lambda) = 0$.

If P and Q are associate, denote by $S(P, Q)$ the set of the maps of the form $Ad(y)$ with $y \in G_{\mathbb{Q}}$ such that $Ad(y)(A_P) = A_Q$.

Scattering Matrices

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For two such associate parabolic subgroups P and Q, we have

$$E_{P|Q}(x,\phi,\lambda) = \sum_{s\in S(P,Q)} e^{(\tau_Q+s\lambda)(H_Q(\gamma x))}(C_{P|Q}(s,\lambda)\phi)(\pi_Q(\gamma x))$$

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Note that $C_{P|Q}(s,\lambda)$ is a meromorphic family of operators

mapping $\phi \in L^2(S_P)$ with eigenvalue μ to $\phi_1 \in L^2(S_Q)$. It can be shown that ϕ_1 is in fact an eigenfunction of the Laplacian acting on S_Q.

Reduction theory of S

Choose a set of representatives $\{P_1, P_2, ..., P_n\}$ of Γ-conjugacy classes of rational parabolic subgroups of G as well as a set of representatives $\{Q_1, Q_2, ..., Q_m\}$ of Γ-conjugacy classes of maximal rational parabolic subgroups. Let $\mathfrak{a} = \oplus_i \mathfrak{a}_{Q_i}$ with $\mathfrak{a}_{Q_i} = Lie(A_{Q_i})$. Then S admits the following decompostion,

$$S = S^T \cup \bigsqcup_{i=1}^{n} S^T_{P_i}$$

Where, S^T is a compact submanifold with corners of codimension zero in S and $S^T_{P_i}$(called the Siegel end associated to P_i at height T) are of the form $\pi(\omega_i \times A_{P_i,T})$ with $T \in \mathfrak{a}$, $T \gg 0$. The $\omega_i \subset N_{P_i} M_{P_i}$ are compact and $A_{P_i,T}$ are shifted Weyl chambers in A_{P_i} appropriately defined depending on T.

Scattering geodesics in S

Scattering geodesics in S

Definition

A geodesic $\gamma(t)$ in S is a scattering geodesic if it is eventually minimizing in both directions, in the sense that there exists $t_1 < t_2 \in \mathbb{R}$ such that $\gamma|_{(-\infty,t_1]}$ and $\gamma|_{[t_2,\infty)}$ are both isometric embedding onto its image.

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Note that since, such a scattering geodesic runs off to infinity in both direction, it spends a finite amount of time in the compact core of S. Define the **Sojourn time** associated to such a scattering geodesic as the time this geodesic spends in the compact core starting from the moment it entered till the instance it left the compact core.

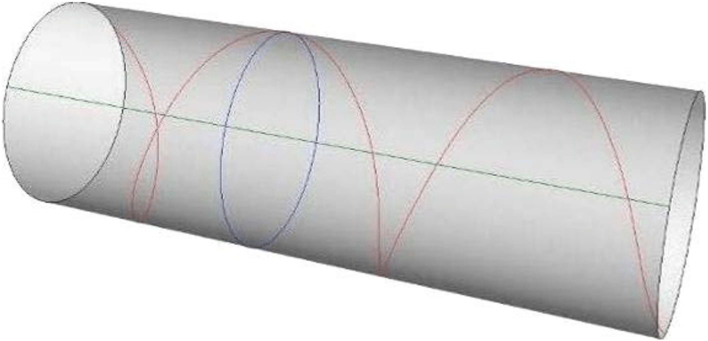

Figure: Geodesics on a cylinder

Scattering geodesics in S

Scattering geodesics in S

Theorem (JZ)

Assume that the $\mathbb{Q}$-rank of S is equal to one. Let $\sigma(t)$ be a scattering geodesic in S between the ends associated with two rational parabolic subgroups Q_1 and Q_2. Then $\sigma(t)$ lies in a smooth family of scattering geodesics of the same sojourn time parametrized by a common finite covering space X_{12} of the boundary locally symmetric spaces X^{Q_1} and X^{Q_2} and the set of sojourn times of all scattering geodesics forms a discrete sequence of points in $\mathbb{R}$ of finite multiplicities.

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Theorem (JZ)

Assume that the $\mathbb{Q}$*-rank of* S *is equal to one. Let* $C_{ij}(\lambda)$ *denote the scattering matrix associated to the ith and j-th end of* S*. Then we have* $\{Singular\ Support(C_{ij}(\lambda))\} = \mathcal{T}$*, where* $\mathcal{T}$ *is the set of Sojorun times associated to scattering geodesic running between the ith and jth end of* S*.*

Scattering Flats in S

Scattering Flats in S

Definition

Let $\mathfrak{a}$ be an abelian subalgebra of $\mathfrak{g}$, denote by Σ the set of roots corresponding to the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$, then a scattering flat is a flat submanifold F of $\Gamma\backslash X$ of dimension equal to rank($\mathfrak{a}$), given by a smooth immersion $\Psi : \mathfrak{a} \longrightarrow \Gamma\backslash X$, such that for any choice of a full subset of positive roots Σ^+ of Σ with the associated positive chamber $\mathfrak{a}^+$, we have that the restriction of Ψ to the shifted Weyl chamber $\mathfrak{a}^+(\Sigma, H)$ is an isometric embedding into $\Gamma\backslash X$, where $H \in \mathfrak{a}^+$ with $|H| >> 0$ and $\mathfrak{a}^+(\Sigma, H) = \{X \in \mathfrak{a} | \beta(X - H) > 0 \forall \beta \in \Sigma^+\}$

Construction of Scattering Flats

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Let P be a rational parabolic subgroup of G with rational rank p and rational Langlands decomposition $P = M_P N_P A_P$ along with associated boundary symmetric space given by X^P. Further let $\Sigma^+(P, A_P)$ be the set of positive roots associated to the adjoint action of $\mathfrak{a}_P = \mathrm{Lie}(A_P)$ on $n_P = \mathrm{Lie}(N_P)$ and the associated positive chamber $\mathfrak{a}_P^+$. Then for any $(z, n) \in X^P \times N_P$ and $H_1, H_2, ..., H_p \in \mathfrak{a}_P^+$ linearly independent, $\tilde{F} = (z, exp_P(t_1 H_1 + t_2 H_2 + ... t_p H_p), n)$ is a flat submanifold in X and consequently $F = \pi(\tilde{F})$ is a scattering flat in S, where $\pi : X \to S$ is the projection map.

Construction of Scattering Flats

From here on, we will focus on minimal rational parabolic subgroups of G. Given two different minimal rational parabolic subgroups P_1 and P_2, we can always choose a common rational split component A, with respect to such a split component we have the Langlands decomposition $P_i = N_i MA$ where we necessarily have $N_1 \neq N_2$ and they correspond to different chambers in A.

Consider the flat submanifold $\tilde{F}$ of X given in the horospherical coordinates(with respect to P_1), $\tilde{S} = (z, exp(t_1 H_1 + ... + t_n H_n), Id)$ with $t_i \in \mathbb{R}$ and $H_i \in \mathfrak{a}_1^+$, where a_1^+ is the positive chamber in $\mathfrak{a} = Lie(A)$ corresponding to N_1 and $z \in M/(M \cap K)$.

Construction of Scattering Flats

Since the N-component of any point in $\tilde{F}$ is trivial, $\tilde{F}$ would have similar horospherical coordinates(with respect to P_2) given by $\tilde{F} = (z, exp(r_1K_1 + ... + r_nK_n), Id)$ with $r_i \in \mathbb{R}$ and $K_i \in \mathfrak{a}_2^+$, where a_2^+ is the positive chamber in $\mathfrak{a} = Lie(A)$ corresponding to N_2.

So the projection of $\tilde{F}$ in S, given by $\pi(\tilde{F})$ is a scattering flat in S corresponding to the pair $\{P_1, P_2\}$ and scattering between the Siegel ends corresponding to P_1 and P_2.

Main results on Scattering Flats

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Theorem

Let $\Gamma \backslash X$ be a locally symmetric space of $\mathbb{Q}$-rank $q > 1$, for two rational parabolic subgroups Q_1 and Q_2 of G; a scattering flat exists between the associated Siegel ends(which could be the same) if and only if Q_1 and Q_2 are associate. In case Q_1 and Q_2 are associate, the choice of a common split component of Q_1 and Q_2 gives rise to a smooth family of scattering flats between the corresponding Siegel ends parametrized by a common finite cover of S_{Q_1} and S_{Q_2}.

Main results on Scattering Flats

Theorem

Let $\Gamma \backslash X$ be a locally symmetric space of rational rank $q > 1$. Fix two distinct minimal rational parabolic subgroups Q_1 and Q_2 of G. Then, for any $\gamma \in \Gamma$ their exists a common rational split component A_γ of Q_1 and Q_2. Further suppose that with respect to the Langlands decomposition $Q_1 = MA_\gamma N$, γ admits a Bruhat decomposition given by $\gamma = u_2 \gamma_a z w u_1$ with $u_1, u_2 \in N$, $\gamma_a \in A_\gamma$, $z \in M$ and $w \in N_K(A_\gamma)$. Then the common split component A_γ gives rise to a family of q dimensional scattering flats between the Siegel ends corresponding to Q_1 and Q_2 with a common sojourn vector which is defined to be $log(\gamma_a) \in Lie(A_\gamma)$.

Review of Lie algebra and parabolic subgroups of $SL(3, \mathbb{R})$

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We will denote by $\mathfrak{g}$ the lie algebra of $SL(3, \mathbb{R})$ consisting of 3 by 3 traceless matrices with real entries. Let H denote the cartan subalgebra consisting of diagonal matrices in $\mathfrak{g}$, we will identify H with the subspace of $\mathbb{R}^3$ given by $\{(h_1, h_2, h_3) \in \mathbb{R}^3 | h_1 + h_2 + h_3 = 0\}$.

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Denote by $\alpha_{ij} : H \longrightarrow \mathbb{R}$ as the functionals defined by $\alpha_{ij}(h_1, h_2, h_3) = h_i - h_j$. Further let $\mathfrak{g}^{\beta} = \{Y \in \mathfrak{g} | [h, Y] = \beta(h)Y \; \forall h \in H\}$ where $\beta : H \longrightarrow \mathbb{R}$ is a nonzero linear functional.

Review of Lie algebra and parabolic subgroups of $SL(3, \mathbb{R})$

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The lie algebra $\mathfrak{g}$ admits the following root space decomposition,

$$\mathfrak{g} = H \oplus_{i \neq j} g^{\alpha_{ij}} \tag{3}$$

where E_{ij} denote the three by three matrix all of whose entires are zero, except for the ij-th entry which is 1.

Review of Lie algebra and parabolic subgroups of $SL(3, \mathbb{R})$

The lie algebra $\mathfrak{g}$ admits the following root space decomposition,

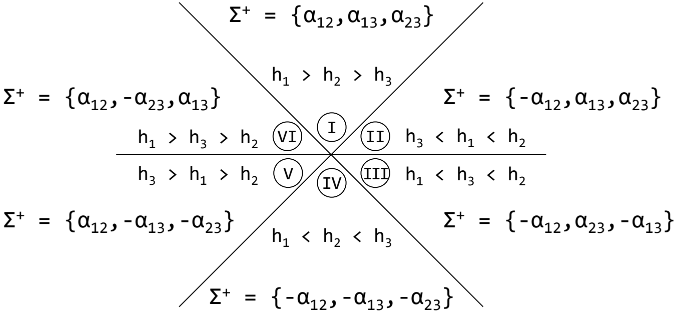
$$\mathfrak{g} = H \oplus_{i \neq j} g^{\alpha_{ij}} \tag{3}$$

where E_{ij} denote the three by three matrix all of whose entires are zero, except for the ij-th entry which is 1.

Note that for the adjoint action of H on $\mathfrak{g}$, $\Sigma^{++} = \{\alpha_{12}, \alpha_{23}\}$ serves as a set of simple roots and the corresponding set of positive roots is given by $\Sigma^{+} = \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}$. We further define $\tau : H \longrightarrow \mathbb{R}$, to be half the sum of the three positive roots, more explicitly for $h = (h_1, h_2, h_3) \in H$, we have $\tau(h) = h_1 - h_3$. Associated to this root space decomposition , we have the Weyl group $\mathcal{W} = S_3$, which acts on the Cartan subalgebra H by permuting coordinates.

Review of Lie algebra and parabolic subgroups of $SL(3,\mathbb{R})$

Cartan Subalgebra $H = \left\{(h_1,h_2,h_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} h_i = 0\right\}$



The Weyl group acts transitively on these chambers.

Review of Lie algebra and parabolic subgroups of $SL(3,\mathbb{R})$

Each chamber in H gives rise to a minimal parabolic subgroup in $SL(3,\mathbb{R})$ with a common split component H and any arbitrary minimal parabolic subgroup of $SL(3,\mathbb{R})$ is K-conjugate to one of these six parabolic subgroups where K is $SO(3)$.

Let N be the set of upper triangular unipotent matrices in $SL(3,\mathbb{R})$, A the subgroup of diagonal matrices of $SL(3,\mathbb{R})$ with positive diagonal entries. Then we denote by P_0 the minimal parabolic subgroup of $SL(3,\mathbb{R})$ consisting of upper traingular matrices along with the Langlands decomposition $P_0 = M_0AN$, where $M_0 = \{\pm Id_{3\times 3}\}$.

Apart from P_0, $SL(3,\mathbb{R})$ has two other standard maximal parabolic subgroups, which we denote by P_1 and P_2, any other parabolic subgroup is conjugate to one of these standard ones.

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

Let $G = SL(3,\mathbb{R})$ and $K = SO(3)$, we pick the arithmetic subgroup of G given by $\Gamma = SL(3,\mathbb{Z})$. We will look at two dimensional scattering flats in the locally symmetric space $S = \Gamma\backslash G/K$. Since, there is only one Γ-conjugacy class of minimal parabolic subgroups of $SL(3,\mathbb{R})$ with our chosen representative P_0, w.l.o.g we can say that any such scattering flats scatters between the Siegel end associated to P_0 and itself.

Choose $\gamma \in \Gamma - P_0$. Then γ admits a Bruhat decomposition with respect to P_0 given by $\gamma = u_2 z \gamma_a w u_1$ with $u_1, u_2 \in N$, $\gamma_a \in A$ and $z \in M_0, w \in S_3$

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

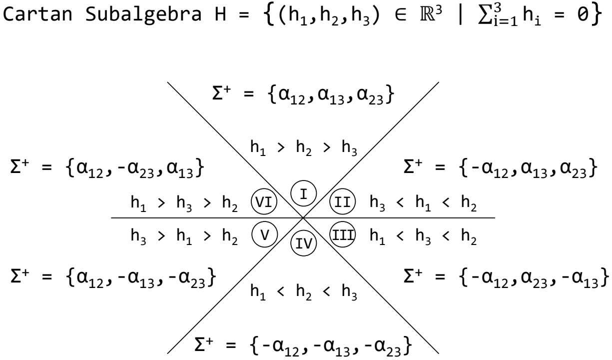
Note that such a γ gives rise to a family of scattering flats with a common associated sojourn vector given by $\log(\gamma_a) \in H$. Each scattering flat in this family scatters between the Siegel ends corresponding to the parabolic subgroups P_0 and $P_1 = \gamma P_0 \gamma^{-1}$.

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

Note that such a γ gives rise to a family of scattering flats with a common associated sojourn vector given by $\log(\gamma_a) \in H$. Each scattering flat in this family scatters between the Siegel ends corresponding to the parabolic subgroups P_0 and $P_1 = \gamma P_0 \gamma^{-1}$.

In case P_0 and P_1 correspond to adjacent chambers we can choose a maximal parabolic subgroup Q containing both P_0, P_1 along with Langlands decomposition $Q = M_Q A_Q N_Q$ and two $\mathbb{Q}$ rank one rational parabolic subgroups Q_0, Q_1 of M_Q, such that the boundary symmetric space X^Q can be naturally identified with the upper half plane $\mathbb{H}$ and the associated boundary locally space S_Q with the quotient $SL(2,\mathbb{Z})\backslash\mathbb{H}$.

chamber I corresponds to P_0,chamber II gives parabolic subgroup $P_1 = wP_0w^{-1}$, with $w = (12) \in S_3$.



one can find a maximal parabolic subgroup P of $SL(3,\mathbb{R})$ which contains both P_0 and P_1. The family of scattering flats in S associated to the pair $\{P_0, P_1\}$ then projects onto a family of scattering geodesic in $SL(2,\mathbb{Z})\backslash\mathbb{H}$ with a common sojourn time whose A_P component corresponds to the common wall shared between chambers I and II.

Scattering Flats in $SL(3, \mathbb{Z}) \backslash SL(3, \mathbb{R}) / SO(3)$

Theorem

If two minimal parabolic subgroups P_0 and P_1 correspond to adjacent chambers(with respect to a common split component), then the family of scattering flats in S corresponding to the pair $\{P_0, P_1\}$ projects onto a family of scattering geodesics in $SL(2, Z) \backslash \mathbb{H}$ running between ends corresponding to Q_0 and Q_1 with a common sojourn time given by $|log(\gamma_a)|$, where $| \bullet |$ is the norm on the Lie algebra $\mathfrak{a}$ associated to the Killing form.

For future reference, denote the set of sojourn times associated to scattering geodesics running between the single cusp of $SL(2, Z) \backslash \mathbb{H}$ as $\mathcal{T}$.

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

We will now look at the rank two scattering matrices associated to S. Since, there is only one association class of minimal parabolic subgroups of $SL(3,\mathbb{R})$, these matrices only depend on $w \in S_3$ and $\lambda \in H^* \otimes_{\mathbb{R}} \mathbb{C}$ and denoted by $C(w,\lambda)$.
Also denote by $C(s)$ the scattering matrix associated with the unique cusp of $SL(2,\mathbb{Z})\backslash\mathbb{H}$.

Scattering Flats in $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$

Let $w = (12) \in S_3$, with $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in H^* \otimes_{\mathbb{R}} \mathbb{C}$ with $Re(\lambda) >> 0$ and $a, b \in \{1,2,3\}$ chosen such that $a < b$ and $w(a) > w(b)$, we have for $Im(\lambda_a - \lambda_b + 1) \leq -3$ and $\tilde{s} = (1/2)(\lambda_a - \lambda_b + 1)$,

$$C(w,\lambda) = F(\tilde{s}) \sum_{T \in \mathcal{T}} e^{-T(1/2 + i\tilde{s})}$$

where,

$$F(\tau) = \int_{-\infty}^{\infty} (1 + w^2)^{-(1/2 + i\tau)} dw$$

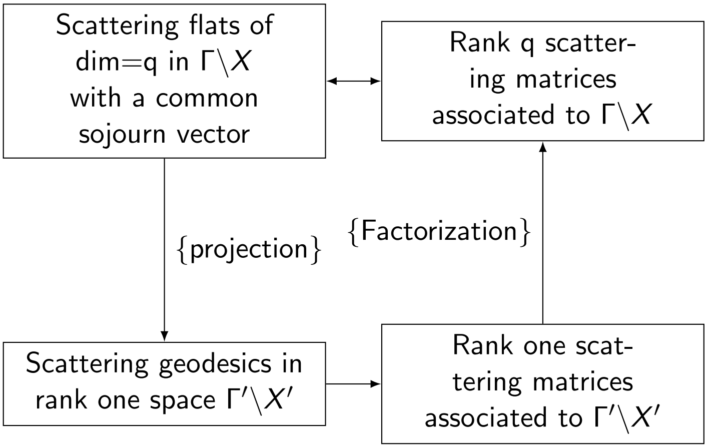


Figure: Correspondence between scattering flats and scattering matrices.

Thank you!

Main References

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