Bernstein components and Hecke algebras for *p*-adic groups

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Maarten Solleveld, Radboud Universiteit Bernstein components and Hecke algebras for

G: reductive group over a non-archimedean local field $F \operatorname{Rep}(G)$: category of smooth complex G-representations

Bernstein decomposition

Direct product of categories $\operatorname{Rep}(G) = \prod_{\mathfrak{s}} \operatorname{Rep}(G)^{\mathfrak{s}}$ where \mathfrak{s} is determined by a supercuspidal representation σ of a Levi subgroup M of G

We suppose that M and σ are given

Questions

- What does Rep(G)^{\$} look like?
- Is it the module category of an explicit algebra?
- Can one classify $Irr(G)^{\mathfrak{s}} = Irr(G) \cap Rep(G)^{\mathfrak{s}}$?
- Can one describe tempered/unitary/square-integrable representations in Rep(G)⁵?

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I. Bernstein components and a rough version of the new results

- P = MU: parabolic subgroup of G with Levi factor M
- $I_P^G : \operatorname{Rep}(M) \to \operatorname{Rep}(P) \to \operatorname{Rep}(G)$: normalized parabolic induction

Definition

For $\pi \in \operatorname{Irr}(G)$:

- π is supercuspidal if it does not occur in $I_P^G(\sigma)$ for any proper parabolic subgroup P of G and any $\sigma \in Irr(M)$
- Supercuspidal support Sc(π): a pair (M, σ) with σ ∈ Irr(M), such that π is a constituent of I_P^G(σ) and M is minimal for this property

 $X_{\mathrm{nr}}(M)$: group of unramified characters $M \to \mathbb{C}^{\times}$ $\mathcal{O} \subset \mathrm{Irr}(M)$: an $X_{\mathrm{nr}}(M)$ -orbit of supercuspidal irreps $\mathfrak{s} = [M, \mathcal{O}]$: *G*-association class of (M, \mathcal{O})

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 $Irr(G)^{\mathfrak{s}} = \{\pi \in Irr(G) : Sc(\pi) \in [M, \mathcal{O}]\}$ Rep(G)^{\$} = { $\pi \in Rep(G)$: all irreducible subquotients of π lie in $Irr(G)^{\mathfrak{s}}$ }

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I: an Iwahori subgroup of G

$$\operatorname{Rep}(G)' = \left\{ (\pi, V) \in \operatorname{Rep}(G) : V \text{ is generated by } V' \right\}$$

The foremost example of a Bernstein component, for $\mathfrak{s} = [M, X_{nr}(M)]$ where M is a minimal Levi subgroup of G

Theorem (Borel, Iwahori-Matsumoto, Morris)

 $\mathcal{H}(G, I) := C_c(I \setminus G/I)$ with the convolution product

- $\operatorname{Rep}(G)^{I}$ is equivalent with $\operatorname{Mod}(\mathcal{H}(G, I))$
- $\mathcal{H}(G, I)$ is isomorphic with an affine Hecke algebra

When G is F-split, M = T and these affine Hecke algebras are understood very well from Kazhdan–Lusztig

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$$N_G(M)$$
 acts on $\operatorname{Rep}(M)$ by $(g \cdot \sigma)(m) = \sigma(g^{-1}mg)$
 $W(M, \mathcal{O}) = \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$

 $\mathbb{C}[\mathcal{O}]$: ring of regular functions on the complex torus $\mathcal O$

Theorem (Bernstein, 1984)

The centre of $\operatorname{Rep}(G)^{\mathfrak{s}}$ is $\mathbb{C}[\mathcal{O}]^{W(M,\mathcal{O})}$

 $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})] := \mathbb{C}[\mathcal{O}] \otimes_{\mathbb{C}} \mathbb{C}[W(M, \mathcal{O})]$ with multiplication from $W(M, \mathcal{O})$ -action on \mathcal{O} :

$$(f \otimes w)(f' \otimes w') = f w(f') \otimes ww'$$

Main result (first rough version)

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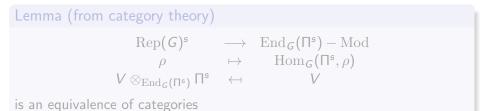
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 $\Pi^{\mathfrak{s}}: \text{ progenerator of } \operatorname{Rep}(G)^{\mathfrak{s}} \\ \text{ so } \Pi^{\mathfrak{s}} \in \operatorname{Rep}(G)^{\mathfrak{s}} \text{ is finitely generated, projective and } \operatorname{Hom}_{G}(\Pi^{\mathfrak{s}}, \rho) \neq 0 \text{ for every } \rho \in \operatorname{Rep}(G)^{\mathfrak{s}} \setminus \{0\}$



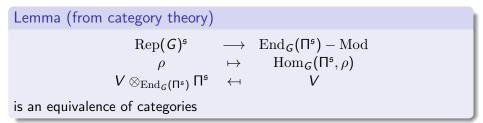
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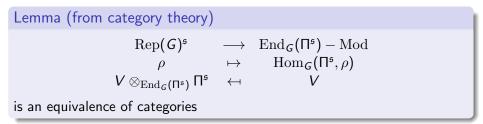
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II. The structure of supercuspidal Bernstein components

based on work of Roche

Underlying tori

 $\sigma \in \operatorname{Irr}(G) \text{ supercuspidal} \\ \mathcal{O} = \{ \sigma \otimes \chi : \chi \in X_{\operatorname{nr}}(G) \} \\ \operatorname{Covering} X_{\operatorname{nr}}(G) \to \mathcal{O} : \chi \mapsto \sigma \otimes \chi \end{cases}$

 $X_{\mathrm{nr}}(G,\sigma) := \{\chi \in X_{\mathrm{nr}}(G) : \sigma \otimes \chi \cong \sigma\}$, a finite group $X_{\mathrm{nr}}(G)/X_{\mathrm{nr}}(G,\sigma) \to \mathcal{O}$ is bijective, this makes \mathcal{O} a complex algebraic torus (as variety)

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 G^1 : subgroup of G generated by all compact subgroups $(\sigma, E) \in Irr(G)$ supercuspidal (σ_1, E_1) : irreducible G^1 -subrepresentation of (σ, E)

Lemma (Bernstein)

 $\operatorname{ind}_{G^1}^G(\sigma_1, E_1)$ is a progenerator of $\operatorname{Rep}(G)^{\mathfrak{s}}$, with $\mathfrak{s} = [G, \mathcal{O}] = [G, X_{\operatorname{nr}}(G)\sigma]$

Some *G*-endomorphisms of $\operatorname{ind}_{G^1}^G(\sigma_1, E_1)$

 $\mathbb{C}[\mathcal{O}] \cong \mathbb{C}[G_{\sigma}/G^{1}] \text{ with } G^{1} \subset G_{\sigma} \subset G$ $b_{g} \in \mathbb{C}[\mathcal{O}] \text{ acts as } (b_{g} \cdot v)(g') = \sigma(g^{-1})v(gg')$

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A multiplicity one conjecture

Conjecture

Let $(\sigma, E) \in Irr(G)$ be supercuspidal Then every irreducible G^1 -subrepresentation of σ has multiplicity one in $\sigma|_{G^1}$

Roche proved the conjecture when:

- dim $Z(G) \leq 1$ or
- G is quasi-split (e.g. a torus) or
- G is a direct products of such groups

In the remainder of the talk this conjecture is assumed If the conjecture fails, there still are similar (almost equally strong) results, but more difficult to state

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Structure of $\operatorname{Rep}(G)^{\mathfrak{s}}$

 E_1 irreducible G^1 -subrep of supercuspidal $E \in Irr(G)$ $\mathfrak{s} = [G, \mathcal{O}]_G, \mathcal{O} = X_{\mathrm{nr}}(G)\sigma$

Theorem (Roche) $\operatorname{End}_{\mathcal{G}}(\operatorname{ind}_{\mathcal{G}^1}^{\mathcal{G}}(\mathcal{E}_1)) \cong \mathbb{C}[\mathcal{O}]$

Corollary

 $\operatorname{Rep}(G)^{\mathfrak{s}} \cong \mathbb{C}[\mathcal{O}]$ -Mod

The finite length subcategory $\operatorname{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}}$ is equivalent with the category of coherent sheaves on the algebraic variety \mathcal{O}

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III. Structure of non-supercuspidal Bernstein components

Motivated by work of Heiermann for classical p-adic groups

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P = MU: parabolic subgroup of G, $(\sigma, E) \in Irr(M)$ supercuspidal $\mathcal{O} = X_{nr}(M)\sigma$, $\mathfrak{s} = [M, \mathcal{O}]$, E_1 irreducible M^1 -subrep of E

Theorem (Bernstein)

 $\Pi^{\mathfrak{s}} := I_{P}^{\mathcal{G}}(\operatorname{ind}_{M^{1}}^{\mathcal{M}}(E_{1})) \text{ is a progenerator of } \operatorname{Rep}(\mathcal{G})^{\mathfrak{s}}$ In particular $\operatorname{Rep}(\mathcal{G})^{\mathfrak{s}} \cong \operatorname{End}_{\mathcal{G}}(\Pi^{\mathfrak{s}})\text{-}\mathsf{Mod}$

This is deep, it relies on second adjointness

Via I_P^G , $\mathbb{C}[\mathcal{O}]$ embeds in $\operatorname{End}_G(\Pi^{\mathfrak{s}})$

Lemma

 $\rho \in \operatorname{Irr}(G)^{\mathfrak{s}}$. Suppose that the $\operatorname{End}_{G}(\Pi^{\mathfrak{s}})$ -module $\operatorname{Hom}_{G}(\Pi^{\mathfrak{s}}, \rho)$ has a $\mathbb{C}[\mathcal{O}]$ -weight σ' . Then ρ has supercuspidal support (M, σ') .

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Example:
$$SL_2(F)$$

 $M = T, \sigma = \text{triv}, \mathcal{O} = X_{nr}(T) \cong \mathbb{C}^{\times}$
 $W(G, T) = \{1, s_{\alpha}\}$

Harish-Chandra's intertwining operator

 $I_{s_{\alpha}}(\chi): I_{P}^{G}(\chi) \to I_{P}^{G}(\chi^{-1}), \quad f \mapsto \left[g \mapsto \int_{U_{-\alpha}} f(us_{\alpha}g) du\right]$ rational as function of $\chi \in X_{\mathrm{nr}}(T)$

$$\operatorname{End}_{G}(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[X_{\mathrm{nr}}(\mathcal{T})]}{\otimes} \mathbb{C}(X_{\mathrm{nr}}(\mathcal{T})) = \mathbb{C}(X_{\mathrm{nr}}(\mathcal{T})) \rtimes \mathbb{C}[1, J_{s_{\alpha}}]$$

where $J_{s_{\alpha}}$ comes from $I_{s_{\alpha}}$, acting as $\chi \mapsto \chi^{-1}$ on $X_{\mathrm{nr}}(\mathcal{T})$, $J_{s_{\alpha}}^{2} = 1$

Singularities of $J_{s_{\alpha}}$

at $\chi \in X_{nr}(T)$ with $\chi(\alpha^{\vee}(\text{uniformizer of } F)) = q_F^{\pm 1}$ For these χ : $I_P^G(\chi)$ is reducible

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Example: $SL_2(F)$ $M = T \sigma = triv (2 - X)(T)$

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Singularities of $J_{s_{\alpha}}$ at $\chi \in X_{nr}(T)$ with $\chi(\alpha^{\vee}(\text{uniformizer of } F)) = q_F^{\pm 1}$ For these χ : $I_P^G(\chi)$ is reducible

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- \bullet acts naturally on ${\cal O}$ by automorphisms of algebraic varieties
- there is a root system $\Sigma_{\mathcal{O}} \subset \Sigma(G, Z(M)^{\circ})$ with $W(\Sigma_{\mathcal{O}}) \subset W(M, \mathcal{O})$
- W(M, O) = W(Σ_O) ⋊ C[R(O)], where R(O) is the stabilizer of the set of positive roots Σ⁺_O

Example

$$G = GL_n(F), \quad M = T = GL_1(F)^n, \quad \sigma = \tau^{\boxtimes n}$$

$$\Sigma_{\mathcal{O}} = \Sigma(G, T) = A_{n-1}, \quad W(M, \mathcal{O}) = W(\Sigma_{\mathcal{O}}) = S_n$$

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$$G = SL_n(F), \quad M = T \cap SL_n(F), \quad \sigma = 1 \boxtimes \tau \boxtimes \ldots \boxtimes \tau^{n-1} \text{ with}$$

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Structure of $\operatorname{End}_{\mathcal{G}}(\Pi^{\mathfrak{s}})$

$\mathbb{C}(\mathcal{O}){:}$ quotient field of $\mathbb{C}[\mathcal{O}]{,}$ rational functions on \mathcal{O}

Main result (weak version)

There exists an algebra isomorphism

$$\operatorname{End}_{G}(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[\mathcal{O}]}{\otimes} \mathbb{C}(\mathcal{O}) \cong \bigoplus_{r \in R(\mathcal{O})} \left(\mathbb{C}(\mathcal{O}) \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})] \right) T_{r}$$

Here $\mathbb{C}(\mathcal{O}) \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})] = \mathbb{C}(\mathcal{O}) \otimes_{\mathbb{C}} \mathbb{C}[W(\Sigma_{\mathcal{O}})]$ with multiplication coming from the action of $W(\Sigma_{\mathcal{O}})$ on \mathcal{O} $T_r T_{r'} = \natural(r, r') T_{rr'}$ where \natural is a 2-cocycle with values in $\mathbb{C}[\mathcal{O}]^{\times}$

In some examples the 2-cocycle is nontrivial

This result only says something about $\operatorname{Rep}(G)^{\mathfrak{s}} \cong \operatorname{End}_{G}(\Pi^{\mathfrak{s}})$ -Mod outside the tricky points of the cuspidal support variety \mathcal{O}

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IV. Links with affine Hecke algebras

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Sketch of an affine Hecke algebra

• Start with $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]$

• For every simple reflection $s_{\alpha} \in W(\Sigma_{\mathcal{O}})$, replace the relation $(s_{\alpha} + 1)(s_{\alpha} - 1) = 0$ in $\mathbb{C}[W(\Sigma_{\mathcal{O}})]$ by

$$(\mathcal{T}_{s_{lpha}}+1)(\mathcal{T}_{s_{lpha}}-q_{F}^{\lambda(lpha)})=0 \quad ext{for some } \lambda(lpha)\in\mathbb{R}_{\geq0}$$

- Adjust the multiplication relations between C[O] and the T_{s_α} (as in the Bernstein presentation)
- This gives an affine Hecke algebra *H̃*(*O*) with the same underlying vector space C[*O*] ⊗ C[*W*(Σ_{*O*})]
- $\mathbb{C}[\mathcal{O}]$ is still a maximal commutative subalgebra

Sketch of an affine Hecke algebra

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- Adjust the multiplication relations between $\mathbb{C}[\mathcal{O}]$ and the $T_{s_{\alpha}}$ (as in the Bernstein presentation)
- This gives an affine Hecke algebra $\tilde{\mathcal{H}}(\mathcal{O})$ with the same underlying vector space $\mathbb{C}[\mathcal{O}] \otimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]$
- $\mathbb{C}[\mathcal{O}]$ is still a maximal commutative subalgebra

Localization

We analyse the category of those $\operatorname{End}_{G}(\Pi^{\mathfrak{s}})$ -modules, all whose $\mathbb{C}[\mathcal{O}]$ -weights lie in a specified subset $U \subset \mathcal{O}$ These are related to $\tilde{\mathcal{H}}(\mathcal{O})$ -modules with $\mathbb{C}[\mathcal{O}]$ -weights in U

Subset of \mathcal{O} to localize on: Fix one $\sigma' \in \mathcal{O}$ and define $U = W(M, \mathcal{O}) \{ \sigma' \otimes \chi : \chi \in X^+_{nr}(M) \}$ $X^+_{nr}(M) := \operatorname{Hom}(M/M^1, \mathbb{R}_{>0}) \subset X_{nr}(M)$

Advantage:

on U we can replace $\natural : R(\mathcal{O})^2 \to \mathbb{C}[\mathcal{O}]^{\times}$ by $\natural_U : R(\mathcal{O})^2 \to \mathbb{C}^{\times}$

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on U we can replace $\natural : R(\mathcal{O})^2 \to \mathbb{C}[\mathcal{O}]^{\times}$ by $\natural_U : R(\mathcal{O})^2 \to \mathbb{C}^{\times}$

 $\begin{array}{l} {\cal G} \colon \mbox{ reductive } p\mbox{-adic group} \\ {\cal O} = \{\sigma\otimes\chi:\chi\in X_{\rm nr}({\cal M})\}, {\mathfrak s} = [{\cal M},{\cal O}] \\ {\Pi}^{\mathfrak s} \colon \mbox{ progenerator of Bernstein block } {\rm Rep}({\cal G})^{\mathfrak s} \end{array}$

$$\begin{split} & U = W(M,\mathcal{O})\sigma' X^+_{\mathrm{nr}}(M) \subset \mathcal{O} \\ & \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U] \text{ constructed by modification o} \\ & \left(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]\right) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U] \\ & (\text{with certain specific parameters } q_F^{\lambda(\alpha)}) \end{split}$$

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There are equivalences between the following categories:

- $\{\pi \in \operatorname{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} : \operatorname{Sc}(\pi) \subset (M, U)\}$ (fl : finite length)
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Under a mild condition (conjecturally always fulfilled) on the 2-cocycle \natural of $R(\mathcal{O})$ involved in $\operatorname{End}_{G}(\Pi^{\mathfrak{s}})$:

Corollary

There is an equivalence of categories between $\operatorname{Rep}(G)^{\mathfrak{s}} \quad \text{and} \quad \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural] - \operatorname{Mod}$

Extras

The above equivalences of categories respect parabolic induction, temperedness and square-integrability of representations

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V. Classification of irreducible representations in $\operatorname{Rep}(G)^{\mathfrak{s}}$

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Representations of affine Hecke algebras

- From an equivalence Rep_{fl}(G)^{\$} ≅ H
 (O) ⋊ C[R(O), ↓] Mod_{fl}, Irr(G)^{\$} can be determined in terms of affine Hecke algebras
- The irreps of an affine Hecke algebra (also when extended with $R(\mathcal{O})$) are known in principle, but their classification is involved

Replacing q_F by 1 in affine Hecke algebras

- $q_F = 1$ -version of $\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural] : \mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]$
- Its representation theory is easy, with Clifford theory

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There is a canonical bijection

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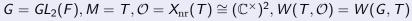
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Example



$$\begin{array}{ll} q_{F} > 1 & q_{F} = 1 \\ \widetilde{\mathcal{H}}(\mathcal{O}) & \mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G,T)] \\ I_{B}^{G}(\sigma) & \operatorname{ind}_{\mathbb{C}[\mathcal{O}]}^{\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G,T)]}(\sigma) \\ I_{B}^{G}(\chi,\chi) & (\chi,\chi) \otimes \operatorname{triv}_{W(G,T)} \\ \operatorname{St} \otimes \chi \circ \det & (\chi,\chi) \otimes \operatorname{sign}_{W(G,T)} \\ \chi \circ \det & \operatorname{ind}_{\mathbb{C}[\mathcal{O}]}^{\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G,T)]}(q_{F}^{1/2}\chi) \end{array}$$

$$\begin{aligned} \sigma &= (\sigma_1, \sigma_2) \in X_{\mathrm{nr}}(T), \quad \sigma_1 \sigma_2^{-1} \notin \{1, q_F, q_F^{-1}\} \\ \chi &\in X_{\mathrm{nr}}(\mathsf{GL}_1(F)), \quad \chi \circ \det \cong I_B^G(q_F^{1/2}\chi, q_F^{-1/2}\chi) / (\mathrm{St} \otimes \chi \circ \det) \end{aligned}$$

Maarten Solleveld, Radboud Universiteit

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Theorem

There exist canonical bijections between the following sets

- $Irr(G)^{s}$
- $\operatorname{Irr}(\operatorname{End}_{G}(\Pi^{\mathfrak{s}}))$
- $\operatorname{Irr}(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural])$ under a mild condition on \natural
- $\{(\sigma',\rho): \sigma' \in \mathcal{O}, \rho \in \operatorname{Irr}(\mathbb{C}[\operatorname{Stab}_{W(M,\mathcal{O})}(\sigma'), \natural_U])\}/W(M,\mathcal{O})$

The last item is also known as a twisted extended quotient

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Summary

For an arbitrary Bernstein block $\operatorname{Rep}(G)^{\mathfrak{s}}$ of a reductive *p*-adic group *G*:

- Rep_{fl}(G)^s is equivalent with the category of finite length modules of an extended affine Hecke algebra *H̃*(*O*) ⋊ ℂ[*R*(*O*), *𝔅*], whose *q_F* = 1-form is ℂ[*O*] ⋊ ℂ[*W*(*M*, *O*), *𝔅*]
- Upon tensoring with C(O) over C[O], or upon taking irreducible representations, Rep(G)^{\$} becomes equivalent with C[O] ⋊ C[W(M, O), \$] Mod

Questions / open problems

- Can one use the above to study unitarity of G-representations?
- Can the parameters q_F^{λ(α)} of H̃(O) be described in terms of σ or O? Are the λ(α) integers?
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