Bernstein components and Hecke algebras for p-adic groups

Maarten Solleveld
Radboud Universiteit Nijmegen

2 February 2021

G: reductive group over a non-archimedean local field F
$\mathrm{Rep}(G)$: category of smooth complex G-representations

Bernstein decomposition

Direct product of categories $\mathrm{Rep}(G) = \prod_{\mathfrak{s}} \mathrm{Rep}(G)^{\mathfrak{s}}$
where $\mathfrak{s}$ is determined by a supercuspidal representation σ of a Levi subgroup M of G

We suppose that M and σ are given

Questions

- What does $\mathrm{Rep}(G)^{\mathfrak{s}}$ look like?
- Is it the module category of an explicit algebra?
- Can one classify $\mathrm{Irr}(G)^{\mathfrak{s}} = \mathrm{Irr}(G) \cap \mathrm{Rep}(G)^{\mathfrak{s}}$?
- Can one describe tempered/unitary/square-integrable representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$?

G: reductive group over a non-archimedean local field F
$\mathrm{Rep}(G)$: category of smooth complex G-representations

Bernstein decomposition

Direct product of categories $\mathrm{Rep}(G) = \prod_{\mathfrak{s}} \mathrm{Rep}(G)^{\mathfrak{s}}$
where $\mathfrak{s}$ is determined by a supercuspidal representation σ of a Levi subgroup M of G

We suppose that M and σ are given

Questions

- What does $\mathrm{Rep}(G)^{\mathfrak{s}}$ look like?
- Is it the module category of an explicit algebra?
- Can one classify $\mathrm{Irr}(G)^{\mathfrak{s}} = \mathrm{Irr}(G) \cap \mathrm{Rep}(G)^{\mathfrak{s}}$?
- Can one describe tempered/unitary/square-integrable representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$?

G: reductive group over a non-archimedean local field F
$\mathrm{Rep}(G)$: category of smooth complex G-representations

Bernstein decomposition

Direct product of categories $\mathrm{Rep}(G) = \prod_{\mathfrak{s}} \mathrm{Rep}(G)^{\mathfrak{s}}$
where $\mathfrak{s}$ is determined by a supercuspidal representation σ of a Levi subgroup M of G

We suppose that M and σ are given

Questions

- What does $\mathrm{Rep}(G)^{\mathfrak{s}}$ look like?
- Is it the module category of an explicit algebra?
- Can one classify $\mathrm{Irr}(G)^{\mathfrak{s}} = \mathrm{Irr}(G) \cap \mathrm{Rep}(G)^{\mathfrak{s}}$?
- Can one describe tempered/unitary/square-integrable representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$?

I. Bernstein components and a rough version of the new results

Bernstein components

$P = MU$: parabolic subgroup of G with Levi factor M
$I_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(P) \to \mathrm{Rep}(G)$: normalized parabolic induction

Definition

For $\pi \in \mathrm{Irr}(G)$:

- π is supercuspidal if it does not occur in $I_P^G(\sigma)$ for any proper parabolic subgroup P of G and any $\sigma \in \mathrm{Irr}(M)$
- Supercuspidal support $\mathrm{Sc}(\pi)$: a pair (M, σ) with $\sigma \in \mathrm{Irr}(M)$, such that π is a constituent of $I_P^G(\sigma)$ and M is minimal for this property

$X_{\mathrm{nr}}(M)$: group of unramified characters $M \to \mathbb{C}^\times$
$\mathcal{O} \subset \mathrm{Irr}(M)$: an $X_{\mathrm{nr}}(M)$-orbit of supercuspidal irreps
$\mathfrak{s} = [M, \mathcal{O}]$: G-association class of $(M, \mathcal{O})$

Definition

$\mathrm{Irr}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Irr}(G) : \mathrm{Sc}(\pi) \in [M, \mathcal{O}]\}$
$\mathrm{Rep}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Rep}(G) :$ all irreducible subquotients of π lie in $\mathrm{Irr}(G)^{\mathfrak{s}}\}$

Bernstein components

$P = MU$: parabolic subgroup of G with Levi factor M

$I_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(P) \to \mathrm{Rep}(G)$: normalized parabolic induction

Definition

For $\pi \in \mathrm{Irr}(G)$:

- π is supercuspidal if it does not occur in $I_P^G(\sigma)$ for any proper parabolic subgroup P of G and any $\sigma \in \mathrm{Irr}(M)$
- Supercuspidal support $\mathrm{Sc}(\pi)$: a pair (M, σ) with $\sigma \in \mathrm{Irr}(M)$, such that π is a constituent of $I_P^G(\sigma)$ and M is minimal for this property

$X_{\mathrm{nr}}(M)$: group of unramified characters $M \to \mathbb{C}^\times$

$\mathcal{O} \subset \mathrm{Irr}(M)$: an $X_{\mathrm{nr}}(M)$-orbit of supercuspidal irreps

$\mathfrak{s} = [M, \mathcal{O}]$: G-association class of $(M, \mathcal{O})$

Definition

$\mathrm{Irr}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Irr}(G) : \mathrm{Sc}(\pi) \in [M, \mathcal{O}]\}$

$\mathrm{Rep}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Rep}(G) :$ all irreducible subquotients of π lie in $\mathrm{Irr}(G)^{\mathfrak{s}}\}$

Bernstein components

$P = MU$: parabolic subgroup of G with Levi factor M
$I_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(P) \to \mathrm{Rep}(G)$: normalized parabolic induction

Definition

For $\pi \in \mathrm{Irr}(G)$:

- π is supercuspidal if it does not occur in $I_P^G(\sigma)$ for any proper parabolic subgroup P of G and any $\sigma \in \mathrm{Irr}(M)$
- Supercuspidal support $\mathrm{Sc}(\pi)$: a pair (M, σ) with $\sigma \in \mathrm{Irr}(M)$, such that π is a constituent of $I_P^G(\sigma)$ and M is minimal for this property

$X_{\mathrm{nr}}(M)$: group of unramified characters $M \to \mathbb{C}^\times$
$\mathcal{O} \subset \mathrm{Irr}(M)$: an $X_{\mathrm{nr}}(M)$-orbit of supercuspidal irreps
$\mathfrak{s} = [M, \mathcal{O}]$: G-association class of $(M, \mathcal{O})$

Definition

$\mathrm{Irr}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Irr}(G) : \mathrm{Sc}(\pi) \in [M, \mathcal{O}]\}$
$\mathrm{Rep}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Rep}(G) :$ all irreducible subquotients of π lie in $\mathrm{Irr}(G)^{\mathfrak{s}}\}$

Bernstein components

$P = MU$: parabolic subgroup of G with Levi factor M
$I_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(P) \to \mathrm{Rep}(G)$: normalized parabolic induction

Definition

For $\pi \in \mathrm{Irr}(G)$:

- π is supercuspidal if it does not occur in $I_P^G(\sigma)$ for any proper parabolic subgroup P of G and any $\sigma \in \mathrm{Irr}(M)$
- Supercuspidal support $\mathrm{Sc}(\pi)$: a pair (M, σ) with $\sigma \in \mathrm{Irr}(M)$, such that π is a constituent of $I_P^G(\sigma)$ and M is minimal for this property

$X_{\mathrm{nr}}(M)$: group of unramified characters $M \to \mathbb{C}^\times$
$\mathcal{O} \subset \mathrm{Irr}(M)$: an $X_{\mathrm{nr}}(M)$-orbit of supercuspidal irreps
$\mathfrak{s} = [M, \mathcal{O}]$: G-association class of $(M, \mathcal{O})$

Definition

$\mathrm{Irr}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Irr}(G) : \mathrm{Sc}(\pi) \in [M, \mathcal{O}]\}$
$\mathrm{Rep}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Rep}(G) :$ all irreducible subquotients of π lie in $\mathrm{Irr}(G)^{\mathfrak{s}}\}$

Iwahori-spherical component

I: an Iwahori subgroup of G

$$\mathrm{Rep}(G)^I = \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\}$$

The foremost example of a Bernstein component,
for $\mathfrak{s} = [M, X_{\mathrm{nr}}(M)]$ where M is a minimal Levi subgroup of G

Theorem (Borel, Iwahori–Matsumoto, Morris)

$\mathcal{H}(G, I) := C_c(I \setminus G / I)$ with the convolution product

- $\mathrm{Rep}(G)^I$ is equivalent with $\mathrm{Mod}(\mathcal{H}(G, I))$
- $\mathcal{H}(G, I)$ is isomorphic with an affine Hecke algebra

When G is F-split, $M = T$ and these affine Hecke algebras are understood very well from Kazhdan–Lusztig

Iwahori-spherical component

I: an Iwahori subgroup of G

$$\mathrm{Rep}(G)^I = \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\}$$

The foremost example of a Bernstein component,
for $\mathfrak{s} = [M, X_{\mathrm{nr}}(M)]$ where M is a minimal Levi subgroup of G

Theorem (Borel, Iwahori–Matsumoto, Morris)

$\mathcal{H}(G, I) := C_c(I \setminus G / I)$ with the convolution product

- $\mathrm{Rep}(G)^I$ is equivalent with $\mathrm{Mod}(\mathcal{H}(G, I))$
- $\mathcal{H}(G, I)$ is isomorphic with an affine Hecke algebra

When G is F-split, $M = T$ and these affine Hecke algebras are understood very well from Kazhdan–Lusztig

Iwahori-spherical component

I: an Iwahori subgroup of G

$$\mathrm{Rep}(G)^I = \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\}$$

The foremost example of a Bernstein component,
for $\mathfrak{s} = [M, X_{\mathrm{nr}}(M)]$ where M is a minimal Levi subgroup of G

Theorem (Borel, Iwahori–Matsumoto, Morris)

$\mathcal{H}(G, I) := C_c(I \setminus G / I)$ with the convolution product

- $\mathrm{Rep}(G)^I$ is equivalent with $\mathrm{Mod}(\mathcal{H}(G, I))$
- $\mathcal{H}(G, I)$ is isomorphic with an affine Hecke algebra

When G is F-split, $M = T$ and these affine Hecke algebras are understood very well from Kazhdan–Lusztig

Iwahori-spherical component

I: an Iwahori subgroup of G

$$\mathrm{Rep}(G)^I = \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\}$$

The foremost example of a Bernstein component,
for $\mathfrak{s} = [M, X_{\mathrm{nr}}(M)]$ where M is a minimal Levi subgroup of G

Theorem (Borel, Iwahori–Matsumoto, Morris)

$\mathcal{H}(G, I) := C_c(I \setminus G / I)$ with the convolution product

- $\mathrm{Rep}(G)^I$ is equivalent with $\mathrm{Mod}(\mathcal{H}(G, I))$
- $\mathcal{H}(G, I)$ is isomorphic with an affine Hecke algebra

When G is F-split, $M = T$ and these affine Hecke algebras are understood very well from Kazhdan–Lusztig

Centre of a Bernstein component

$N_G(M)$ acts on $\mathrm{Rep}(M)$ by $(g \cdot \sigma)(m) = \sigma(g^{-1}mg)$

$$W(M, \mathcal{O}) = \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$$

$\mathbb{C}[\mathcal{O}]$: ring of regular functions on the complex torus $\mathcal{O}$

Theorem (Bernstein, 1984)

The centre of $\mathrm{Rep}(G)^{\mathfrak{s}}$ is $\mathbb{C}[\mathcal{O}]^{W(M,\mathcal{O})}$

$\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})] := \mathbb{C}[\mathcal{O}] \otimes_{\mathbb{C}} \mathbb{C}[W(M, \mathcal{O})]$ with multiplication from $W(M, \mathcal{O})$-action on $\mathcal{O}$:

$$(f \otimes w)(f' \otimes w') = f\, w(f') \otimes ww'$$

Main result (first rough version)

$\mathrm{Rep}(G)^{\mathfrak{s}}$ looks like $\mathrm{Mod}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})]\big)$

Centre of a Bernstein component

$N_G(M)$ acts on $\mathrm{Rep}(M)$ by $(g \cdot \sigma)(m) = \sigma(g^{-1} m g)$

$$W(M, \mathcal{O}) = \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$$

$\mathbb{C}[\mathcal{O}]$: ring of regular functions on the complex torus $\mathcal{O}$

Theorem (Bernstein, 1984)

The centre of $\mathrm{Rep}(G)^{\mathfrak{s}}$ is $\mathbb{C}[\mathcal{O}]^{W(M,\mathcal{O})}$

$\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})] := \mathbb{C}[\mathcal{O}] \otimes_{\mathbb{C}} \mathbb{C}[W(M, \mathcal{O})]$ with multiplication from $W(M, \mathcal{O})$-action on $\mathcal{O}$:

$$(f \otimes w)(f' \otimes w') = f\, w(f') \otimes ww'$$

Main result (first rough version)

$\mathrm{Rep}(G)^{\mathfrak{s}}$ looks like $\mathrm{Mod}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})]\big)$

Centre of a Bernstein component

$N_G(M)$ acts on $\mathrm{Rep}(M)$ by $(g \cdot \sigma)(m) = \sigma(g^{-1} m g)$

$$W(M, \mathcal{O}) = \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$$

$\mathbb{C}[\mathcal{O}]$: ring of regular functions on the complex torus $\mathcal{O}$

Theorem (Bernstein, 1984)

The centre of $\mathrm{Rep}(G)^{\mathfrak{s}}$ is $\mathbb{C}[\mathcal{O}]^{W(M,\mathcal{O})}$

$\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})] := \mathbb{C}[\mathcal{O}] \otimes_{\mathbb{C}} \mathbb{C}[W(M, \mathcal{O})]$ with multiplication from $W(M, \mathcal{O})$-action on $\mathcal{O}$:

$$(f \otimes w)(f' \otimes w') = f\, w(f') \otimes ww'$$

Main result (first rough version)

$\mathrm{Rep}(G)^{\mathfrak{s}}$ looks like $\mathrm{Mod}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})]\big)$

Centre of a Bernstein component

$N_G(M)$ acts on $\mathrm{Rep}(M)$ by $(g \cdot \sigma)(m) = \sigma(g^{-1} m g)$

$$W(M, \mathcal{O}) = \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$$

$\mathbb{C}[\mathcal{O}]$: ring of regular functions on the complex torus $\mathcal{O}$

Theorem (Bernstein, 1984)

The centre of $\mathrm{Rep}(G)^{\mathfrak{s}}$ is $\mathbb{C}[\mathcal{O}]^{W(M,\mathcal{O})}$

$\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})] := \mathbb{C}[\mathcal{O}] \otimes_{\mathbb{C}} \mathbb{C}[W(M, \mathcal{O})]$ with multiplication from $W(M, \mathcal{O})$-action on $\mathcal{O}$:

$$(f \otimes w)(f' \otimes w') = f\, w(f') \otimes ww'$$

Main result (first rough version)

$\mathrm{Rep}(G)^{\mathfrak{s}}$ looks like $\mathrm{Mod}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})]\big)$

Approach with progenerators

$\Pi^{\mathfrak{s}}$: progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$
so $\Pi^{\mathfrak{s}} \in \mathrm{Rep}(G)^{\mathfrak{s}}$ is finitely generated, projective and $\mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho) \neq 0$ for every $\rho \in \mathrm{Rep}(G)^{\mathfrak{s}} \setminus \{0\}$

Lemma (from category theory)

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \longrightarrow & \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod} \\ \rho & \mapsto & \mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho) \\ V \otimes_{\mathrm{End}_G(\Pi^{\mathfrak{s}})} \Pi^{\mathfrak{s}} & \leftarrow\!\shortmid & V \end{array}$$

is an equivalence of categories

Goal

Investigate the structure and the representation theory of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$, for a suitable progenerator $\Pi^{\mathfrak{s}}$ of $\mathrm{Rep}(G)^{\mathfrak{s}}$
Draw consequences for $\mathrm{Rep}(G)^{\mathfrak{s}}$

Approach with progenerators

$\Pi^{\mathfrak{s}}$: progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$
so $\Pi^{\mathfrak{s}} \in \mathrm{Rep}(G)^{\mathfrak{s}}$ is finitely generated, projective and $\mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho) \neq 0$ for every $\rho \in \mathrm{Rep}(G)^{\mathfrak{s}} \setminus \{0\}$

Lemma (from category theory)

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \longrightarrow & \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod} \\ \rho & \mapsto & \mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho) \\ V \otimes_{\mathrm{End}_G(\Pi^{\mathfrak{s}})} \Pi^{\mathfrak{s}} & \leftarrow\!\shortmid & V \end{array}$$

is an equivalence of categories

Goal

Investigate the structure and the representation theory of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$, for a suitable progenerator $\Pi^{\mathfrak{s}}$ of $\mathrm{Rep}(G)^{\mathfrak{s}}$
Draw consequences for $\mathrm{Rep}(G)^{\mathfrak{s}}$

Approach with progenerators

$\Pi^{\mathfrak{s}}$: progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$
so $\Pi^{\mathfrak{s}} \in \mathrm{Rep}(G)^{\mathfrak{s}}$ is finitely generated, projective and $\mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho) \neq 0$ for every $\rho \in \mathrm{Rep}(G)^{\mathfrak{s}} \setminus \{0\}$

Lemma (from category theory)

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \longrightarrow & \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod} \\ \rho & \mapsto & \mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho) \\ V \otimes_{\mathrm{End}_G(\Pi^{\mathfrak{s}})} \Pi^{\mathfrak{s}} & \leftarrow & V \end{array}$$

is an equivalence of categories

Goal

Investigate the structure and the representation theory of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$, for a suitable progenerator $\Pi^{\mathfrak{s}}$ of $\mathrm{Rep}(G)^{\mathfrak{s}}$
Draw consequences for $\mathrm{Rep}(G)^{\mathfrak{s}}$

II. The structure of supercuspidal Bernstein components

based on work of Roche

Underlying tori

$\sigma \in \mathrm{Irr}(G)$ supercuspidal
$\mathcal{O} = \{\sigma \otimes \chi : \chi \in X_{\mathrm{nr}}(G)\}$
Covering $X_{\mathrm{nr}}(G) \to \mathcal{O} : \chi \mapsto \sigma \otimes \chi$

$X_{\mathrm{nr}}(G, \sigma) := \{\chi \in X_{\mathrm{nr}}(G) : \sigma \otimes \chi \cong \sigma\}$, a finite group
$X_{\mathrm{nr}}(G)/X_{\mathrm{nr}}(G, \sigma) \to \mathcal{O}$ is bijective, this makes $\mathcal{O}$ a complex algebraic torus (as variety)

Underlying tori

$\sigma \in \mathrm{Irr}(G)$ supercuspidal
$\mathcal{O} = \{\sigma \otimes \chi : \chi \in X_{\mathrm{nr}}(G)\}$
Covering $X_{\mathrm{nr}}(G) \to \mathcal{O} : \chi \mapsto \sigma \otimes \chi$

$X_{\mathrm{nr}}(G, \sigma) := \{\chi \in X_{\mathrm{nr}}(G) : \sigma \otimes \chi \cong \sigma\}$, a finite group
$X_{\mathrm{nr}}(G)/X_{\mathrm{nr}}(G, \sigma) \to \mathcal{O}$ is bijective, this makes $\mathcal{O}$ a complex algebraic torus (as variety)

A progenerator

G^1: subgroup of G generated by all compact subgroups
$(\sigma, E) \in \mathrm{Irr}(G)$ supercuspidal
(σ_1, E_1): irreducible G^1-subrepresentation of (σ, E)

Lemma (Bernstein)

$\mathrm{ind}_{G^1}^G(\sigma_1, E_1)$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$, with
$\mathfrak{s} = [G, \mathcal{O}] = [G, X_{\mathrm{nr}}(G)\sigma]$

Some G-endomorphisms of $\mathrm{ind}_{G^1}^G(\sigma_1, E_1)$

$\mathbb{C}[\mathcal{O}] \cong \mathbb{C}[G_\sigma / G^1]$ with $G^1 \subset G_\sigma \subset G$
$b_g \in \mathbb{C}[\mathcal{O}]$ acts as $(b_g \cdot v)(g') = \sigma(g^{-1}) v(gg')$

A progenerator

G^1: subgroup of G generated by all compact subgroups
$(\sigma, E) \in \mathrm{Irr}(G)$ supercuspidal
(σ_1, E_1): irreducible G^1-subrepresentation of (σ, E)

Lemma (Bernstein)

$\mathrm{ind}_{G^1}^G(\sigma_1, E_1)$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$, with
$\mathfrak{s} = [G, \mathcal{O}] = [G, X_{\mathrm{nr}}(G)\sigma]$

Some G-endomorphisms of $\mathrm{ind}_{G^1}^G(\sigma_1, E_1)$

$\mathbb{C}[\mathcal{O}] \cong \mathbb{C}[G_\sigma / G^1]$ with $G^1 \subset G_\sigma \subset G$
$b_g \in \mathbb{C}[\mathcal{O}]$ acts as $(b_g \cdot v)(g') = \sigma(g^{-1}) v(gg')$

A progenerator

G^1: subgroup of G generated by all compact subgroups
$(\sigma, E) \in \mathrm{Irr}(G)$ supercuspidal
(σ_1, E_1): irreducible G^1-subrepresentation of (σ, E)

Lemma (Bernstein)

$\mathrm{ind}_{G^1}^{G}(\sigma_1, E_1)$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$, with
$\mathfrak{s} = [G, \mathcal{O}] = [G, X_{\mathrm{nr}}(G)\sigma]$

Some G-endomorphisms of $\mathrm{ind}_{G^1}^{G}(\sigma_1, E_1)$

$\mathbb{C}[\mathcal{O}] \cong \mathbb{C}[G_\sigma / G^1]$ with $G^1 \subset G_\sigma \subset G$
$b_g \in \mathbb{C}[\mathcal{O}]$ acts as $(b_g \cdot v)(g') = \sigma(g^{-1}) v(gg')$

A multiplicity one conjecture

Conjecture

Let $(\sigma, E) \in \mathrm{Irr}(G)$ be supercuspidal
Then every irreducible G^1-subrepresentation of σ has multiplicity one in $\sigma|_{G^1}$

Roche proved the conjecture when:

- $\dim Z(G) \leq 1$ or
- G is quasi-split (e.g. a torus) or
- G is a direct products of such groups

In the remainder of the talk this conjecture is assumed
If the conjecture fails, there still are similar (almost equally strong) results, but more difficult to state

A multiplicity one conjecture

Conjecture

Let $(\sigma, E) \in \mathrm{Irr}(G)$ be supercuspidal
Then every irreducible G^1-subrepresentation of σ has multiplicity one in $\sigma|_{G^1}$

Roche proved the conjecture when:

- $\dim Z(G) \leq 1$ or
- G is quasi-split (e.g. a torus) or
- G is a direct products of such groups

In the remainder of the talk this conjecture is assumed
If the conjecture fails, there still are similar (almost equally strong) results, but more difficult to state

A multiplicity one conjecture

Conjecture

Let $(\sigma, E) \in \mathrm{Irr}(G)$ be supercuspidal
Then every irreducible G^1-subrepresentation of σ has multiplicity one in $\sigma|_{G^1}$

Roche proved the conjecture when:

- $\dim Z(G) \leq 1$ or
- G is quasi-split (e.g. a torus) or
- G is a direct products of such groups

In the remainder of the talk this conjecture is assumed
If the conjecture fails, there still are similar (almost equally strong) results, but more difficult to state

Structure of $\mathrm{Rep}(G)^{\mathfrak{s}}$

E_1 irreducible G^1-subrep of supercuspidal $E \in \mathrm{Irr}(G)$
$\mathfrak{s} = [G, \mathcal{O}]_G, \mathcal{O} = X_{\mathrm{nr}}(G)\sigma$

Theorem (Roche)

$\mathrm{End}_G\big(\mathrm{ind}_{G^1}^G(E_1)\big) \cong \mathbb{C}[\mathcal{O}]$

Corollary

$\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathbb{C}[\mathcal{O}]$-Mod

The finite length subcategory $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}}$ is equivalent with the category of coherent sheaves on the algebraic variety $\mathcal{O}$

III. Structure of non-supercuspidal Bernstein components

Motivated by work of Heiermann for classical p-adic groups

A progenerator

$P = MU$: parabolic subgroup of G, $(\sigma, E) \in \mathrm{Irr}(M)$ supercuspidal
$\mathcal{O} = X_{\mathrm{nr}}(M)\sigma$, $\quad \mathfrak{s} = [M, \mathcal{O}]$, $\quad E_1$ irreducible M^1-subrep of E

Theorem (Bernstein)

$\Pi^{\mathfrak{s}} := I_P^G \big(\mathrm{ind}_{M^1}^M (E_1)\big)$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$
In particular $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi^{\mathfrak{s}})$-Mod

This is deep, it relies on second adjointness

Via I_P^G, $\mathbb{C}[\mathcal{O}]$ embeds in $\mathrm{End}_G(\Pi^{\mathfrak{s}})$

Lemma

$\rho \in \mathrm{Irr}(G)^{\mathfrak{s}}$. Suppose that the $\mathrm{End}_G(\Pi^{\mathfrak{s}})$-module $\mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho)$ has a $\mathbb{C}[\mathcal{O}]$-weight σ'.
Then ρ has supercuspidal support (M, σ').

A progenerator

$P = MU$: parabolic subgroup of G, $(\sigma, E) \in \mathrm{Irr}(M)$ supercuspidal
$\mathcal{O} = X_{\mathrm{nr}}(M)\sigma$, $\quad \mathfrak{s} = [M, \mathcal{O}]$, $\quad E_1$ irreducible M^1-subrep of E

Theorem (Bernstein)

$\Pi^{\mathfrak{s}} := I_P^G \big(\mathrm{ind}_{M^1}^M (E_1)\big)$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$
In particular $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi^{\mathfrak{s}})$-Mod

This is deep, it relies on second adjointness

Via I_P^G, $\mathbb{C}[\mathcal{O}]$ embeds in $\mathrm{End}_G(\Pi^{\mathfrak{s}})$

Lemma

$\rho \in \mathrm{Irr}(G)^{\mathfrak{s}}$. Suppose that the $\mathrm{End}_G(\Pi^{\mathfrak{s}})$-module $\mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho)$ has a $\mathbb{C}[\mathcal{O}]$-weight σ'.
Then ρ has supercuspidal support (M, σ').

A progenerator

$P = MU$: parabolic subgroup of G, $(\sigma, E) \in \mathrm{Irr}(M)$ supercuspidal
$\mathcal{O} = X_{\mathrm{nr}}(M)\sigma$, $\quad \mathfrak{s} = [M, \mathcal{O}]$, $\quad E_1$ irreducible M^1-subrep of E

Theorem (Bernstein)

$\Pi^{\mathfrak{s}} := I_P^G \left(\mathrm{ind}_{M^1}^M (E_1)\right)$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$
In particular $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi^{\mathfrak{s}})$-Mod

This is deep, it relies on second adjointness

Via I_P^G, $\mathbb{C}[\mathcal{O}]$ embeds in $\mathrm{End}_G(\Pi^{\mathfrak{s}})$

Lemma

$\rho \in \mathrm{Irr}(G)^{\mathfrak{s}}$. Suppose that the $\mathrm{End}_G(\Pi^{\mathfrak{s}})$-module $\mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho)$ has a $\mathbb{C}[\mathcal{O}]$-weight σ'.
Then ρ has supercuspidal support (M, σ').

A progenerator

$P = MU$: parabolic subgroup of G, $(\sigma, E) \in \mathrm{Irr}(M)$ supercuspidal
$\mathcal{O} = X_{\mathrm{nr}}(M)\sigma, \quad \mathfrak{s} = [M, \mathcal{O}], \quad E_1$ irreducible M^1-subrep of E

Theorem (Bernstein)

$\Pi^{\mathfrak{s}} := I_P^G \left(\mathrm{ind}_{M^1}^M (E_1)\right)$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$
In particular $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi^{\mathfrak{s}})$-Mod

This is deep, it relies on second adjointness

Via I_P^G, $\mathbb{C}[\mathcal{O}]$ embeds in $\mathrm{End}_G(\Pi^{\mathfrak{s}})$

Lemma

$\rho \in \mathrm{Irr}(G)^{\mathfrak{s}}$. Suppose that the $\mathrm{End}_G(\Pi^{\mathfrak{s}})$-module $\mathrm{Hom}_G(\Pi^{\mathfrak{s}}, \rho)$ has a $\mathbb{C}[\mathcal{O}]$-weight σ'.
Then ρ has supercuspidal support (M, σ').

Example: $SL_2(F)$

$M = T, \sigma = \text{triv}, \mathcal{O} = X_{\mathrm{nr}}(T) \cong \mathbb{C}^\times$
$W(G,T) = \{1, s_\alpha\}$

Harish-Chandra's intertwining operator

$$I_{s_\alpha}(\chi) : I_P^G(\chi) \to I_P^G(\chi^{-1}), \quad f \mapsto \Big[g \mapsto \int_{U_{-\alpha}} f(u s_\alpha g) \, \mathrm{d}u \Big]$$

rational as function of $\chi \in X_{\mathrm{nr}}(T)$

$$\mathrm{End}_G(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[X_{\mathrm{nr}}(T)]}{\otimes} \mathbb{C}(X_{\mathrm{nr}}(T)) = \mathbb{C}(X_{\mathrm{nr}}(T)) \rtimes \mathbb{C}[1, J_{s_\alpha}]$$

where J_{s_α} comes from I_{s_α}, acting as $\chi \mapsto \chi^{-1}$ on $X_{\mathrm{nr}}(T)$, $J_{s_\alpha}^2 = 1$

Singularities of J_{s_α}

at $\chi \in X_{\mathrm{nr}}(T)$ with $\chi(\alpha^\vee(\text{uniformizer of } F)) = q_F^{\pm 1}$
For these χ: $I_P^G(\chi)$ is reducible

Example: $SL_2(F)$

$M = T, \sigma = \text{triv}, \mathcal{O} = X_{\mathrm{nr}}(T) \cong \mathbb{C}^\times$
$W(G,T) = \{1, s_\alpha\}$

Harish-Chandra's intertwining operator

$$I_{s_\alpha}(\chi) : I_P^G(\chi) \to I_P^G(\chi^{-1}), \quad f \mapsto \Big[g \mapsto \int_{U_{-\alpha}} f(u s_\alpha g) \, \mathrm{d}u \Big]$$

rational as function of $\chi \in X_{\mathrm{nr}}(T)$

$$\mathrm{End}_G(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[X_{\mathrm{nr}}(T)]}{\otimes} \mathbb{C}(X_{\mathrm{nr}}(T)) = \mathbb{C}(X_{\mathrm{nr}}(T)) \rtimes \mathbb{C}[1, J_{s_\alpha}]$$

where J_{s_α} comes from I_{s_α}, acting as $\chi \mapsto \chi^{-1}$ on $X_{\mathrm{nr}}(T)$, $J_{s_\alpha}^2 = 1$

Singularities of J_{s_α}

at $\chi \in X_{\mathrm{nr}}(T)$ with $\chi(\alpha^\vee(\text{uniformizer of } F)) = q_F^{\pm 1}$
For these χ: $I_P^G(\chi)$ is reducible

Example: $SL_2(F)$

$M = T, \sigma = \text{triv}, \mathcal{O} = X_{\mathrm{nr}}(T) \cong \mathbb{C}^\times$
$W(G,T) = \{1, s_\alpha\}$

Harish-Chandra's intertwining operator

$$I_{s_\alpha}(\chi) : I_P^G(\chi) \to I_P^G(\chi^{-1}), \quad f \mapsto \left[g \mapsto \int_{U_{-\alpha}} f(u s_\alpha g)\, \mathrm{d}u \right]$$

rational as function of $\chi \in X_{\mathrm{nr}}(T)$

$$\mathrm{End}_G(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[X_{\mathrm{nr}}(T)]}{\otimes} \mathbb{C}(X_{\mathrm{nr}}(T)) = \mathbb{C}(X_{\mathrm{nr}}(T)) \rtimes \mathbb{C}[1, J_{s_\alpha}]$$

where J_{s_α} comes from I_{s_α}, acting as $\chi \mapsto \chi^{-1}$ on $X_{\mathrm{nr}}(T)$, $J_{s_\alpha}^2 = 1$

Singularities of J_{s_α}

at $\chi \in X_{\mathrm{nr}}(T)$ with $\chi(\alpha^\vee(\text{uniformizer of } F)) = q_F^{\pm 1}$
For these χ: $I_P^G(\chi)$ is reducible

Example: $SL_2(F)$

$M = T, \sigma = \text{triv}, \mathcal{O} = X_{\mathrm{nr}}(T) \cong \mathbb{C}^\times$
$W(G,T) = \{1, s_\alpha\}$

Harish-Chandra's intertwining operator

$$I_{s_\alpha}(\chi) : I_P^G(\chi) \to I_P^G(\chi^{-1}), \quad f \mapsto \Big[g \mapsto \int_{U_{-\alpha}} f(u s_\alpha g)\, \mathrm{d}u \Big]$$

rational as function of $\chi \in X_{\mathrm{nr}}(T)$

$$\mathrm{End}_G(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[X_{\mathrm{nr}}(T)]}{\otimes} \mathbb{C}(X_{\mathrm{nr}}(T)) = \mathbb{C}(X_{\mathrm{nr}}(T)) \rtimes \mathbb{C}[1, J_{s_\alpha}]$$

where J_{s_α} comes from I_{s_α}, acting as $\chi \mapsto \chi^{-1}$ on $X_{\mathrm{nr}}(T)$, $J_{s_\alpha}^2 = 1$

Singularities of J_{s_α}

at $\chi \in X_{\mathrm{nr}}(T)$ with $\chi(\alpha^\vee(\text{uniformizer of } F)) = q_F^{\pm 1}$
For these χ: $I_P^G(\chi)$ is reducible

The Weyl-like group of a Bernstein component

- $W(M,\mathcal{O}) := \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$
- acts naturally on $\mathcal{O}$ by automorphisms of algebraic varieties
- there is a root system $\Sigma_{\mathcal{O}} \subset \Sigma(G, Z(M)^\circ)$ with $W(\Sigma_{\mathcal{O}}) \subset W(M,\mathcal{O})$
- $W(M,\mathcal{O}) = W(\Sigma_{\mathcal{O}}) \rtimes \mathbb{C}[R(\mathcal{O})]$, where $R(\mathcal{O})$ is the stabilizer of the set of positive roots $\Sigma_{\mathcal{O}}^+$

Example

$G = GL_n(F), \quad M = T = GL_1(F)^n, \quad \sigma = \tau^{\boxtimes n}$
$\Sigma_{\mathcal{O}} = \Sigma(G,T) = A_{n-1}, \quad W(M,\mathcal{O}) = W(\Sigma_{\mathcal{O}}) = S_n$

Example

$G = SL_n(F), \quad M = T \cap SL_n(F), \quad \sigma = 1 \boxtimes \tau \boxtimes \ldots \boxtimes \tau^{n-1}$ with
$\tau : F^\times \to \mathbb{C}^\times$ ramified, of order n
$\Sigma_{\mathcal{O}} = \emptyset, \quad W(M,\mathcal{O}) = R(\mathcal{O}) \cong \mathbb{Z}/n\mathbb{Z}$

The Weyl-like group of a Bernstein component

- $W(M,\mathcal{O}) := \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$
- acts naturally on $\mathcal{O}$ by automorphisms of algebraic varieties
- there is a root system $\Sigma_{\mathcal{O}} \subset \Sigma(G, Z(M)^\circ)$ with $W(\Sigma_{\mathcal{O}}) \subset W(M,\mathcal{O})$
- $W(M,\mathcal{O}) = W(\Sigma_{\mathcal{O}}) \rtimes \mathbb{C}[R(\mathcal{O})]$, where $R(\mathcal{O})$ is the stabilizer of the set of positive roots $\Sigma_{\mathcal{O}}^+$

Example

$G = GL_n(F), \quad M = T = GL_1(F)^n, \quad \sigma = \tau^{\boxtimes n}$
$\Sigma_{\mathcal{O}} = \Sigma(G,T) = A_{n-1}, \quad W(M,\mathcal{O}) = W(\Sigma_{\mathcal{O}}) = S_n$

Example

$G = SL_n(F), \quad M = T \cap SL_n(F), \quad \sigma = 1 \boxtimes \tau \boxtimes \ldots \boxtimes \tau^{n-1}$ with
$\tau : F^\times \to \mathbb{C}^\times$ ramified, of order n
$\Sigma_{\mathcal{O}} = \emptyset, \quad W(M,\mathcal{O}) = R(\mathcal{O}) \cong \mathbb{Z}/n\mathbb{Z}$

The Weyl-like group of a Bernstein component

- $W(M, \mathcal{O}) := \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$
- acts naturally on $\mathcal{O}$ by automorphisms of algebraic varieties
- there is a root system $\Sigma_{\mathcal{O}} \subset \Sigma(G, Z(M)^\circ)$ with $W(\Sigma_{\mathcal{O}}) \subset W(M, \mathcal{O})$
- $W(M, \mathcal{O}) = W(\Sigma_{\mathcal{O}}) \rtimes \mathbb{C}[R(\mathcal{O})]$, where $R(\mathcal{O})$ is the stabilizer of the set of positive roots $\Sigma_{\mathcal{O}}^+$

Example

$G = GL_n(F), \quad M = T = GL_1(F)^n, \quad \sigma = \tau^{\boxtimes n}$
$\Sigma_{\mathcal{O}} = \Sigma(G, T) = A_{n-1}, \quad W(M, \mathcal{O}) = W(\Sigma_{\mathcal{O}}) = S_n$

Example

$G = SL_n(F), \quad M = T \cap SL_n(F), \quad \sigma = 1 \boxtimes \tau \boxtimes \ldots \boxtimes \tau^{n-1}$ with
$\tau : F^\times \to \mathbb{C}^\times$ ramified, of order n
$\Sigma_{\mathcal{O}} = \emptyset, \quad W(M, \mathcal{O}) = R(\mathcal{O}) \cong \mathbb{Z}/n\mathbb{Z}$

The Weyl-like group of a Bernstein component

- $W(M, \mathcal{O}) := \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$
- acts naturally on $\mathcal{O}$ by automorphisms of algebraic varieties
- there is a root system $\Sigma_{\mathcal{O}} \subset \Sigma(G, Z(M)^\circ)$ with $W(\Sigma_{\mathcal{O}}) \subset W(M, \mathcal{O})$
- $W(M, \mathcal{O}) = W(\Sigma_{\mathcal{O}}) \rtimes \mathbb{C}[R(\mathcal{O})]$, where $R(\mathcal{O})$ is the stabilizer of the set of positive roots $\Sigma_{\mathcal{O}}^+$

Example

$G = GL_n(F), \quad M = T = GL_1(F)^n, \quad \sigma = \tau^{\boxtimes n}$
$\Sigma_{\mathcal{O}} = \Sigma(G, T) = A_{n-1}, \quad W(M, \mathcal{O}) = W(\Sigma_{\mathcal{O}}) = S_n$

Example

$G = SL_n(F), \quad M = T \cap SL_n(F), \quad \sigma = 1 \boxtimes \tau \boxtimes \ldots \boxtimes \tau^{n-1}$ with
$\tau : F^\times \to \mathbb{C}^\times$ ramified, of order n
$\Sigma_{\mathcal{O}} = \emptyset, \quad W(M, \mathcal{O}) = R(\mathcal{O}) \cong \mathbb{Z}/n\mathbb{Z}$

Structure of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$

$\mathbb{C}(\mathcal{O})$: quotient field of $\mathbb{C}[\mathcal{O}]$, rational functions on $\mathcal{O}$

Main result (weak version)

There exists an algebra isomorphism

$$\mathrm{End}_G(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[\mathcal{O}]}{\otimes} \mathbb{C}(\mathcal{O}) \cong \bigoplus_{r \in R(\mathcal{O})} \Big(\mathbb{C}(\mathcal{O}) \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})] \Big) T_r$$

Here $\mathbb{C}(\mathcal{O}) \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})] = \mathbb{C}(\mathcal{O}) \otimes_{\mathbb{C}} \mathbb{C}[W(\Sigma_{\mathcal{O}})]$ with multiplication coming from the action of $W(\Sigma_{\mathcal{O}})$ on $\mathcal{O}$

$T_r T_{r'} = \natural(r, r') T_{rr'}$ where $\natural$ is a 2-cocycle with values in $\mathbb{C}[\mathcal{O}]^\times$

In some examples the 2-cocycle is nontrivial

This result only says something about $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi^{\mathfrak{s}})$-Mod outside the tricky points of the cuspidal support variety $\mathcal{O}$

Structure of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$

$\mathbb{C}(\mathcal{O})$: quotient field of $\mathbb{C}[\mathcal{O}]$, rational functions on $\mathcal{O}$

Main result (weak version)

There exists an algebra isomorphism

$$\mathrm{End}_G(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[\mathcal{O}]}{\otimes} \mathbb{C}(\mathcal{O}) \cong \bigoplus_{r \in R(\mathcal{O})} \Big(\mathbb{C}(\mathcal{O}) \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})] \Big) T_r$$

Here $\mathbb{C}(\mathcal{O}) \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})] = \mathbb{C}(\mathcal{O}) \otimes_{\mathbb{C}} \mathbb{C}[W(\Sigma_{\mathcal{O}})]$ with multiplication coming from the action of $W(\Sigma_{\mathcal{O}})$ on $\mathcal{O}$
$T_r T_{r'} = \natural(r, r') T_{rr'}$ where $\natural$ is a 2-cocycle with values in $\mathbb{C}[\mathcal{O}]^\times$

In some examples the 2-cocycle is nontrivial

This result only says something about $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi^{\mathfrak{s}})$-Mod outside the tricky points of the cuspidal support variety $\mathcal{O}$

Structure of $\mathrm{End}_G(\Pi^{\mathfrak{s}})$

$\mathbb{C}(\mathcal{O})$: quotient field of $\mathbb{C}[\mathcal{O}]$, rational functions on $\mathcal{O}$

Main result (weak version)

There exists an algebra isomorphism

$$\mathrm{End}_G(\Pi^{\mathfrak{s}}) \underset{\mathbb{C}[\mathcal{O}]}{\otimes} \mathbb{C}(\mathcal{O}) \cong \bigoplus_{r \in R(\mathcal{O})} \Big(\mathbb{C}(\mathcal{O}) \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})] \Big) T_r$$

Here $\mathbb{C}(\mathcal{O}) \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})] = \mathbb{C}(\mathcal{O}) \otimes_{\mathbb{C}} \mathbb{C}[W(\Sigma_{\mathcal{O}})]$ with multiplication coming from the action of $W(\Sigma_{\mathcal{O}})$ on $\mathcal{O}$

$T_r T_{r'} = \natural(r, r') T_{rr'}$ where $\natural$ is a 2-cocycle with values in $\mathbb{C}[\mathcal{O}]^\times$

In some examples the 2-cocycle is nontrivial

This result only says something about $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi^{\mathfrak{s}})$-Mod outside the tricky points of the cuspidal support variety $\mathcal{O}$

IV. Links with affine Hecke algebras

Sketch of an affine Hecke algebra

- Start with $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]$
- For every simple reflection $s_\alpha \in W(\Sigma_{\mathcal{O}})$, replace the relation $(s_\alpha + 1)(s_\alpha - 1) = 0$ in $\mathbb{C}[W(\Sigma_{\mathcal{O}})]$ by

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - q_F^{\lambda(\alpha)}) = 0 \quad \text{for some } \lambda(\alpha) \in \mathbb{R}_{\geq 0}$$

- Adjust the multiplication relations between $\mathbb{C}[\mathcal{O}]$ and the T_{s_α} (as in the Bernstein presentation)
- This gives an affine Hecke algebra $\tilde{\mathcal{H}}(\mathcal{O})$ with the same underlying vector space $\mathbb{C}[\mathcal{O}] \otimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]$
- $\mathbb{C}[\mathcal{O}]$ is still a maximal commutative subalgebra

Sketch of an affine Hecke algebra

- Start with $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]$
- For every simple reflection $s_\alpha \in W(\Sigma_{\mathcal{O}})$, replace the relation $(s_\alpha + 1)(s_\alpha - 1) = 0$ in $\mathbb{C}[W(\Sigma_{\mathcal{O}})]$ by

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - q_F^{\lambda(\alpha)}) = 0 \quad \text{for some } \lambda(\alpha) \in \mathbb{R}_{\geq 0}$$

- Adjust the multiplication relations between $\mathbb{C}[\mathcal{O}]$ and the T_{s_α} (as in the Bernstein presentation)
- This gives an affine Hecke algebra $\tilde{\mathcal{H}}(\mathcal{O})$ with the same underlying vector space $\mathbb{C}[\mathcal{O}] \otimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]$
- $\mathbb{C}[\mathcal{O}]$ is still a maximal commutative subalgebra

Localization

We analyse the category of those $\mathrm{End}_G(\Pi^{\mathfrak{s}})$-modules, all whose $\mathbb{C}[\mathcal{O}]$-weights lie in a specified subset $U \subset \mathcal{O}$
These are related to $\tilde{\mathcal{H}}(\mathcal{O})$-modules with $\mathbb{C}[\mathcal{O}]$-weights in U

Subset of $\mathcal{O}$ to localize on:

Fix one $\sigma' \in \mathcal{O}$ and define

$$U = W(M, \mathcal{O})\{\sigma' \otimes \chi : \chi \in X_{\mathrm{nr}}^+(M)\}$$
$$X_{\mathrm{nr}}^+(M) := \mathrm{Hom}(M/M^1, \mathbb{R}_{>0}) \subset X_{\mathrm{nr}}(M)$$

Advantage:
on U we can replace $\natural : R(\mathcal{O})^2 \to \mathbb{C}[\mathcal{O}]^\times$ by $\natural_U : R(\mathcal{O})^2 \to \mathbb{C}^\times$

Localization

We analyse the category of those $\mathrm{End}_G(\Pi^{\mathfrak{s}})$-modules, all whose $\mathbb{C}[\mathcal{O}]$-weights lie in a specified subset $U \subset \mathcal{O}$
These are related to $\tilde{\mathcal{H}}(\mathcal{O})$-modules with $\mathbb{C}[\mathcal{O}]$-weights in U

Subset of $\mathcal{O}$ to localize on:

Fix one $\sigma' \in \mathcal{O}$ and define

$$U = W(M, \mathcal{O})\{\sigma' \otimes \chi : \chi \in X_{\mathrm{nr}}^+(M)\}$$
$$X_{\mathrm{nr}}^+(M) := \mathrm{Hom}(M/M^1, \mathbb{R}_{>0}) \subset X_{\mathrm{nr}}(M)$$

Advantage:
on U we can replace $\natural : R(\mathcal{O})^2 \to \mathbb{C}[\mathcal{O}]^\times$ by $\natural_U : R(\mathcal{O})^2 \to \mathbb{C}^\times$

Main result

G: reductive p-adic group
$\mathcal{O} = \{\sigma \otimes \chi : \chi \in X_{\mathrm{nr}}(M)\}, \mathfrak{s} = [M, \mathcal{O}]$
$\Pi^{\mathfrak{s}}$: progenerator of Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$

$U = W(M, \mathcal{O})\sigma' X_{\mathrm{nr}}^{+}(M) \subset \mathcal{O}$
$\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U]$ constructed by modification of
$(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U]$
(with certain specific parameters $q_F^{\lambda(\alpha)}$)

Theorem

There are equivalences between the following categories:

- $\{\pi \in \mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} : \mathrm{Sc}(\pi) \subset (M, U)\}$ (fl : finite length)
- $\{V \in \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of V in $U\}$
- $\{\tilde{V} \in \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U] - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of $\tilde{V}$ in $U\}$

Main result

G: reductive p-adic group
$\mathcal{O} = \{\sigma \otimes \chi : \chi \in X_{\mathrm{nr}}(M)\}, \mathfrak{s} = [M, \mathcal{O}]$
$\Pi^{\mathfrak{s}}$: progenerator of Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$

$U = W(M, \mathcal{O})\sigma' X_{\mathrm{nr}}^{+}(M) \subset \mathcal{O}$
$\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U]$ constructed by modification of
$(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(\Sigma_{\mathcal{O}})]) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U]$
(with certain specific parameters $q_F^{\lambda(\alpha)}$)

Theorem

There are equivalences between the following categories:

- $\{\pi \in \mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} : \mathrm{Sc}(\pi) \subset (M, U)\}$ (fl : finite length)
- $\{V \in \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of V in $U\}$
- $\{\tilde{V} \in \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U] - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of $\tilde{V}$ in $U\}$

Main result

Theorem (independent of multiplicity one conjecture)

There are equivalences between the following categories:

- $\{\pi \in \mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} : \mathrm{Sc}(\pi) \subset (M, U)\}$ (fl : finite length)
- $\{V \in \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of V in $U\}$
- $\{\tilde{V} \in \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U] - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of $\tilde{V}$ in $U\}$

Under a mild condition (conjecturally always fulfilled) on the 2-cocycle $\natural$ of $R(\mathcal{O})$ involved in $\mathrm{End}_G(\Pi^{\mathfrak{s}})$:

Corollary

There is an equivalence of categories between

$$\mathrm{Rep}(G)^{\mathfrak{s}} \quad \text{and} \quad \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural] - \mathrm{Mod}$$

Extras

The above equivalences of categories respect parabolic induction, temperedness and square-integrability of representations

Main result

Theorem (independent of multiplicity one conjecture)

There are equivalences between the following categories:

- $\{\pi \in \mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} : \mathrm{Sc}(\pi) \subset (M, U)\}$ (fl : finite length)
- $\{V \in \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of V in $U\}$
- $\{\tilde{V} \in \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U] - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of $\tilde{V}$ in $U\}$

Under a mild condition (conjecturally always fulfilled) on the 2-cocycle $\natural$ of $R(\mathcal{O})$ involved in $\mathrm{End}_G(\Pi^{\mathfrak{s}})$:

Corollary

There is an equivalence of categories between

$$\mathrm{Rep}(G)^{\mathfrak{s}} \quad \text{and} \quad \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural] - \mathrm{Mod}$$

Extras

The above equivalences of categories respect parabolic induction, temperedness and square-integrability of representations

Main result

Theorem (independent of multiplicity one conjecture)

There are equivalences between the following categories:

- $\{\pi \in \mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} : \mathrm{Sc}(\pi) \subset (M, U)\}$ (fl : finite length)
- $\{V \in \mathrm{End}_G(\Pi^{\mathfrak{s}}) - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of V in $U\}$
- $\{\tilde{V} \in \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_U] - \mathrm{Mod}_{\mathrm{fl}} :$ all $\mathbb{C}[\mathcal{O}]$-weights of $\tilde{V}$ in $U\}$

Under a mild condition (conjecturally always fulfilled) on the 2-cocycle $\natural$ of $R(\mathcal{O})$ involved in $\mathrm{End}_G(\Pi^{\mathfrak{s}})$:

Corollary

There is an equivalence of categories between

$$\mathrm{Rep}(G)^{\mathfrak{s}} \quad \text{and} \quad \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural] - \mathrm{Mod}$$

Extras

The above equivalences of categories respect parabolic induction, temperedness and square-integrability of representations

V. Classification of irreducible representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$

Representations of affine Hecke algebras

- From an equivalence $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} \cong \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural] - \mathrm{Mod}_{\mathrm{fl}}$, $\mathrm{Irr}(G)^{\mathfrak{s}}$ can be determined in terms of affine Hecke algebras
- The irreps of an affine Hecke algebra (also when extended with $R(\mathcal{O})$) are known in principle, but their classification is involved

Replacing q_F by 1 in affine Hecke algebras

- $q_F = 1$-version of $\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]$: $\quad \mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]$
- Its representation theory is easy, with Clifford theory

Theorem

There is a canonical bijection

$$\mathrm{Irr}\big(\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]\big) \to \mathrm{Irr}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]\big)$$

It preserves the unitary parts of $\mathbb{C}[\mathcal{O}]^{W(\Sigma_{\mathcal{O}})}$-weights

Representations of affine Hecke algebras

- From an equivalence $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} \cong \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural] - \mathrm{Mod}_{\mathrm{fl}}$, $\mathrm{Irr}(G)^{\mathfrak{s}}$ can be determined in terms of affine Hecke algebras
- The irreps of an affine Hecke algebra (also when extended with $R(\mathcal{O})$) are known in principle, but their classification is involved

Replacing q_F by 1 in affine Hecke algebras

- $q_F = 1$-version of $\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]$: $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]$
- Its representation theory is easy, with Clifford theory

Theorem

There is a canonical bijection

$$\mathrm{Irr}\big(\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]\big) \to \mathrm{Irr}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]\big)$$

It preserves the unitary parts of $\mathbb{C}[\mathcal{O}]^{W(\Sigma_{\mathcal{O}})}$-weights

Representations of affine Hecke algebras

- From an equivalence $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} \cong \tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural] - \mathrm{Mod}_{\mathrm{fl}}$, $\mathrm{Irr}(G)^{\mathfrak{s}}$ can be determined in terms of affine Hecke algebras
- The irreps of an affine Hecke algebra (also when extended with $R(\mathcal{O})$) are known in principle, but their classification is involved

Replacing q_F by 1 in affine Hecke algebras

- $q_F = 1$-version of $\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]$: $\quad \mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]$
- Its representation theory is easy, with Clifford theory

Theorem

There is a canonical bijection

$$\mathrm{Irr}\big(\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]\big) \to \mathrm{Irr}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]\big)$$

It preserves the unitary parts of $\mathbb{C}[\mathcal{O}]^{W(\Sigma_{\mathcal{O}})}$-weights

Irreducible representations of Hecke algebras

Theorem

There is a canonical bijection

$$\mathrm{Irr}\big(\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}),\natural]\big) \to \mathrm{Irr}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M,\mathcal{O}),\natural]\big)$$

Example

$G = GL_2(F), M = T, \mathcal{O} = X_{\mathrm{nr}}(T) \cong (\mathbb{C}^\times)^2, W(T,\mathcal{O}) = W(G,T)$

$q_F > 1$	$q_F = 1$
$\tilde{\mathcal{H}}(\mathcal{O})$	$\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G,T)]$
$I_B^G(\sigma)$	$\mathrm{ind}_{\mathbb{C}[\mathcal{O}]}^{\mathbb{C}[\mathcal{O}]\rtimes\mathbb{C}[W(G,T)]}(\sigma)$
$I_B^G(\chi,\chi)$	$(\chi,\chi)\otimes \mathrm{triv}_{W(G,T)}$
$\mathrm{St}\otimes\chi\circ\det$	$(\chi,\chi)\otimes \mathrm{sign}_{W(G,T)}$
$\chi\circ\det$	$\mathrm{ind}_{\mathbb{C}[\mathcal{O}]}^{\mathbb{C}[\mathcal{O}]\rtimes\mathbb{C}[W(G,T)]}(q_F^{1/2}\chi)$

$\sigma = (\sigma_1,\sigma_2) \in X_{\mathrm{nr}}(T), \quad \sigma_1\sigma_2^{-1} \notin \{1, q_F, q_F^{-1}\}$

$\chi \in X_{\mathrm{nr}}(GL_1(F)), \quad \chi\circ\det \cong I_B^G(q_F^{1/2}\chi, q_F^{-1/2}\chi)/(\mathrm{St}\otimes\chi\circ\det)$

Irreducible representations of Hecke algebras

Theorem

There is a canonical bijection

$$\mathrm{Irr}\big(\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]\big) \to \mathrm{Irr}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]\big)$$

Example

$G = GL_2(F), M = T, \mathcal{O} = X_{\mathrm{nr}}(T) \cong (\mathbb{C}^\times)^2, W(T, \mathcal{O}) = W(G, T)$

$q_F > 1$	$q_F = 1$
$\tilde{\mathcal{H}}(\mathcal{O})$	$\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G, T)]$
$I_B^G(\sigma)$	$\mathrm{ind}_{\mathbb{C}[\mathcal{O}]}^{\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G,T)]}(\sigma)$
$I_B^G(\chi, \chi)$	$(\chi, \chi) \otimes \mathrm{triv}_{W(G,T)}$
$\mathrm{St} \otimes \chi \circ \det$	$(\chi, \chi) \otimes \mathrm{sign}_{W(G,T)}$
$\chi \circ \det$	$\mathrm{ind}_{\mathbb{C}[\mathcal{O}]}^{\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G,T)]}(q_F^{1/2} \chi)$

$\sigma = (\sigma_1, \sigma_2) \in X_{\mathrm{nr}}(T), \quad \sigma_1 \sigma_2^{-1} \notin \{1, q_F, q_F^{-1}\}$

$\chi \in X_{\mathrm{nr}}(GL_1(F)), \quad \chi \circ \det \cong I_B^G(q_F^{1/2}\chi, q_F^{-1/2}\chi)/(\mathrm{St} \otimes \chi \circ \det)$

Irreducible representations of Hecke algebras

Theorem

There is a canonical bijection

$$\mathrm{Irr}\big(\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]\big) \to \mathrm{Irr}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]\big)$$

Example

$G = GL_2(F), M = T, \mathcal{O} = X_{\mathrm{nr}}(T) \cong (\mathbb{C}^\times)^2, W(T, \mathcal{O}) = W(G, T)$

$q_F > 1$	$q_F = 1$
$\tilde{\mathcal{H}}(\mathcal{O})$	$\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G, T)]$
$I_B^G(\sigma)$	$\mathrm{ind}_{\mathbb{C}[\mathcal{O}]}^{\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G,T)]}(\sigma)$
$I_B^G(\chi, \chi)$	$(\chi, \chi) \otimes \mathrm{triv}_{W(G,T)}$
$\mathrm{St} \otimes \chi \circ \det$	$(\chi, \chi) \otimes \mathrm{sign}_{W(G,T)}$
$\chi \circ \det$	$\mathrm{ind}_{\mathbb{C}[\mathcal{O}]}^{\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(G,T)]}(q_F^{1/2} \chi)$

$\sigma = (\sigma_1, \sigma_2) \in X_{\mathrm{nr}}(T), \quad \sigma_1 \sigma_2^{-1} \notin \{1, q_F, q_F^{-1}\}$

$\chi \in X_{\mathrm{nr}}(GL_1(F)), \quad \chi \circ \det \cong I_B^G(q_F^{1/2}\chi, q_F^{-1/2}\chi) / (\mathrm{St} \otimes \chi \circ \det)$

Classification of irreducible representations

Theorem

There exist canonical bijections between the following sets

- $\mathrm{Irr}(G)^{\mathfrak{s}}$
- $\mathrm{Irr}(\mathrm{End}_G(\Pi^{\mathfrak{s}}))$
- $\mathrm{Irr}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M,\mathcal{O}),\natural]\big)$ – under a mild condition on $\natural$
- $\big\{(\sigma',\rho) : \sigma' \in \mathcal{O}, \rho \in \mathrm{Irr}(\mathbb{C}[\mathrm{Stab}_{W(M,\mathcal{O})}(\sigma'),\natural_U])\big\} / W(M,\mathcal{O})$

The last item is also known as a twisted extended quotient

$$(\mathcal{O}//W(M,\mathcal{O}))_\natural$$

The bijection between that and $\mathrm{Irr}(G)^{\mathfrak{s}}$ was conjectured by ABPS

Classification of irreducible representations

Theorem

There exist canonical bijections between the following sets

- $\mathrm{Irr}(G)^{\mathfrak{s}}$
- $\mathrm{Irr}(\mathrm{End}_G(\Pi^{\mathfrak{s}}))$
- $\mathrm{Irr}\big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M,\mathcal{O}),\natural]\big)$ – under a mild condition on $\natural$
- $\big\{(\sigma',\rho) : \sigma' \in \mathcal{O}, \rho \in \mathrm{Irr}(\mathbb{C}[\mathrm{Stab}_{W(M,\mathcal{O})}(\sigma'),\natural_U])\big\} / W(M,\mathcal{O})$

The last item is also known as a twisted extended quotient

$$(\mathcal{O}//W(M,\mathcal{O}))_\natural$$

The bijection between that and $\mathrm{Irr}(G)^{\mathfrak{s}}$ was conjectured by ABPS

Summary

For an arbitrary Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ of a reductive p-adic group G:

- $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}}$ is equivalent with the category of finite length modules of an extended affine Hecke algebra $\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]$, whose $q_F = 1$-form is $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]$
- Upon tensoring with $\mathbb{C}(\mathcal{O})$ over $\mathbb{C}[\mathcal{O}]$, or upon taking irreducible representations, $\mathrm{Rep}(G)^{\mathfrak{s}}$ becomes equivalent with $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural] - \mathrm{Mod}$

Questions / open problems

- Can one use the above to study unitarity of G-representations?
- Can the parameters $q_F^{\lambda(\alpha)}$ of $\tilde{\mathcal{H}}(\mathcal{O})$ be described in terms of σ or $\mathcal{O}$? Are the $\lambda(\alpha)$ integers?
- How to determine the 2-cocycles $\natural$ of $R(\mathcal{O})$?

Summary

For an arbitrary Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ of a reductive p-adic group G:

- $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}}$ is equivalent with the category of finite length modules of an extended affine Hecke algebra $\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]$, whose $q_F = 1$-form is $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]$
- Upon tensoring with $\mathbb{C}(\mathcal{O})$ over $\mathbb{C}[\mathcal{O}]$, or upon taking irreducible representations, $\mathrm{Rep}(G)^{\mathfrak{s}}$ becomes equivalent with $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural] - \mathrm{Mod}$

Questions / open problems

- Can one use the above to study unitarity of G-representations?
- Can the parameters $q_F^{\lambda(\alpha)}$ of $\tilde{\mathcal{H}}(\mathcal{O})$ be described in terms of σ or $\mathcal{O}$? Are the $\lambda(\alpha)$ integers?
- How to determine the 2-cocycles $\natural$ of $R(\mathcal{O})$?

Summary

For an arbitrary Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ of a reductive p-adic group G:

- $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}}$ is equivalent with the category of finite length modules of an extended affine Hecke algebra $\tilde{\mathcal{H}}(\mathcal{O}) \rtimes \mathbb{C}[R(\mathcal{O}), \natural]$, whose $q_F = 1$-form is $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural]$
- Upon tensoring with $\mathbb{C}(\mathcal{O})$ over $\mathbb{C}[\mathcal{O}]$, or upon taking irreducible representations, $\mathrm{Rep}(G)^{\mathfrak{s}}$ becomes equivalent with $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \natural] - \mathrm{Mod}$

Questions / open problems

- Can one use the above to study unitarity of G-representations?
- Can the parameters $q_F^{\lambda(\alpha)}$ of $\tilde{\mathcal{H}}(\mathcal{O})$ be described in terms of σ or $\mathcal{O}$? Are the $\lambda(\alpha)$ integers?
- How to determine the 2-cocycles $\natural$ of $R(\mathcal{O})$?