Motivic Chern classes of Schubert cells and applications.

joint works with Aluffi, Mihalcea, Schurmann;
Lenart, Zainoulline, Zhong

Plan: 0) Chern-Schwartz-MacPherson class in homology

1) Motivic Chern class in K-theory

2) Applications in p-adic group representations.

3) Lenart-Zainoulline-Zhong conjecture

0) Chern-Schwartz-MacPherson class in homology

two functors: $\mathcal{F}(X)$ = constructible functions on an alg. variety $X/\mathbb{C}$
(can be singular).

$H_*(X)$ = Homology of X.

Thm (MacPherson).

$\exists !$ natural transformation

$$c_*: \mathcal{F} \to H_*,$$

s.t. if X is smooth, $c_*(1_X) = c(TX) \cap [X]$.

$f: X \to Y$ proper,

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "f_*"] \arrow[d, "c_*"'] & \mathcal{F}(Y) \arrow[d, dashed, "c_*"] \\
H_*(X) \arrow[r, "f_*"'] & H_*(Y)
\end{tikzcd}$$

Notations:

G s.s. Lie group$/\mathbb{C}$. e.g. $SL(n,\mathbb{C})$

B – Borel subgp. $B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \right\}$

T – max. torus. $T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}$

W – Weyl group $W = S_n$.

$X := G/B$ flag variety, $X = \{ F_\cdot = (F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = \mathbb{C}^n) \mid \dim F_i = i \}$

$w \in W$, $X(w)^\circ := BwB/B$ Schubert cell.

$X(w) := \overline{X(w)^\circ}$ Schubert variety.

Ex. $G = SL(2,\mathbb{C})$, $X = \mathbb{P}^1$,

$$c_*(\mathbb{1}_{X(id)^\circ}) = [X(id)] = [0], \quad c_*(\mathbb{1}_{X(s_\alpha)^\circ}) = c_*(\mathbb{1}_{\mathbb{P}^1}) - c_*(\mathbb{1}_{X(id)^\circ}) = [\mathbb{P}^1] + [\infty]$$

Thm: (Aluffi-Mihalcea-Schurmann-S, 2017)

① $c_*(\mathbb{1}_{X(w)^\circ})$ are permuted by the degenerate Hecke operators

② $i: X \hookrightarrow T^*X$,

$$c_*(\mathbb{1}_{X(w)^\circ}) = i^*[\mathrm{char}(\mathbb{1}_{X(w)^\circ})]\big|_{\hbar=1}$$

equiv. parameter for the natural $\mathbb{C}^*$-dilation action on T^*X.

③ $c_*(\mathbb{1}_{X(w)^\circ})$ is a positive class in $H^*(X)$. (conjectured by Aluffi-Mihalcea).

1) Motivic chern class in K-theory

Definition.

two functors: $Y/\mathbb{C}$, $K^0(Var/Y) := \{[Z \xrightarrow{f} Y]\} / [Z \xrightarrow{f} Y] = [U \xrightarrow{f} Y] + [Z \setminus U \xrightarrow{f} Y]$

$U \subseteq Z$ open.

- $K(Y) := K^0(Coh(Y))$.

Thm (Brasselet-Schürmann-Yokura).

$\exists !$ natural transformation

$MC_y : K^0(Var/-) \to K(-)[y]$, s.t. if Y is smooth,

$MC_y([Y \xrightarrow{id} Y]) = \lambda_y(T^*Y) := \sum_i y^i [\Lambda^i T^*Y]$. Here y is a formal variable.

Remark: $\exists$ equivariant generalizations.

- Flag variety setting.

max torus $T \subseteq B \subseteq G$ $\quad T \curvearrowright X = G/B$. $\quad K_T(X) := K^0(\mathrm{Coh}_T(X))$.

$K_T(pt) = K^0(\mathrm{Coh}_T(pt)) = K^0(\mathrm{Rep}(T)) = \mathbb{Z}[T]$.

Let $MC_y(X(w)^\circ) := MC_y([X(w)^\circ \hookrightarrow X]) \in K_T(X)[y]$.

Ex. $G = SL(2, \mathbb{C})$. $\quad MC_y(X(id)^\circ) = [\mathcal{O}_0]$

$MC_y(X(s_\alpha)^\circ) = MC_y(\mathbb{P}^1) - MC_y(X(id)^\circ) = \lambda_y(T^*\mathbb{P}^1) - [\mathcal{O}_0]$.

Demazure operators.

α_i: simple root, $B \subseteq P_i$ minimal parabolic

e.g. $\alpha_i = \delta_i - \delta_{i+1}$

$$P_i = \left\{ \begin{pmatrix} * & & & * \\ & \ddots & & \\ & & * & \\ & & * & * \\ & & & & \ddots \\ & & & & & * \end{pmatrix} \right\}$$

$(i+1, i)$.

$\pi_i : G/B \to G/P_i$ BGG operator $\partial_i := \pi_i^* \pi_{i*} \circlearrowright K_T(X)$.

then $\partial_i^2 = \partial_i$, braid relation.

$$\partial_i([\mathcal{O}_{X(w)}]) = [\mathcal{O}_{X(ws_i)}] \quad \text{if } ws_i > w.$$

Demazure-Lusztig operator.

$\forall \ \lambda \in X^*(T)$, $\quad \mathcal{L}_\lambda := G \times_B \mathbb{C}_\lambda$

$$\downarrow$$

$$X$$

affine Hecke alg.

Let $T_i = (1+y\mathcal{L}_{\alpha_i})\partial_i - id$, $\quad (T_i+1)(T_i+y) = 0$, $\quad$ Braid relations.

Thm: (Aluffi-Mihalcea-Schürmann-S, 2019).

1). $T_i(\,MC_y(X(w)^\circ)\,) = MC_y(X(ws_i)^\circ)$ $\quad$ if $ws_i > w$.

2) $i: X \hookrightarrow T^*X$, $\quad MC_y(X(w)^\circ) = i^* \, gr[\mathbb{Q}^H_{X(w)^\circ}]$

constant mixed Hodge module.

- Smoothness criterion.

Thm (Kumar). $u \le w \in W$

$$X(w) \text{ smooth at } uB \Leftrightarrow [X(w)]|_u = \prod_{\substack{\alpha>0 \\ ws_\alpha \not\le w}} (-u\alpha) \in H_T^*(pt).$$

Thm: (Aluffi-Mihalcea-Schurmann-S, 2019).

$u \le w \in W$.

$$X(w) \text{ smooth at } uB \Leftrightarrow MC_y(X(w))|_u = \prod_{\substack{\alpha>0 \\ us_\alpha \not\le w}} (1-e^{u\alpha}) \cdot \prod_{\substack{\alpha>0 \\ us_\alpha \le w}} (1+ye^{u\alpha})$$

($\Rightarrow$ use property of $MC_y(X(w)^\circ)$,

$\Leftarrow$ $y=0$, $MC_y(X(w)^\circ)|_{y=0} = [\mathcal{O}_{X(w)}(-\partial X(w))]$, take Chern character).

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2) Applications in p-adic group representations.

- Bump-Nakasuji-Naruse conj.

F non-archimedean local field. $\mathcal{O}_F \subseteq F$, k_F = residue field $= \mathbb{F}_q$, a finite field.

$G^\vee$ = Langlands dual group $/F$., $T^\vee \subseteq B^\vee \subseteq G^\vee$, I = Iwahori-subgroup, $I \subseteq G^\vee(\mathcal{O}_F)$

$$\begin{array}{ccc} I & \subseteq & G^\vee(\mathcal{O}_F) \\ \downarrow & & \downarrow \\ B^\vee(k_F) & \subseteq & G^\vee(k_F) \end{array}$$

τ — an unramified char. of $T^\vee$ ($\Leftrightarrow \tau \in T$)

Principal series $\mathrm{Ind}_{B^\vee(F)}^{G^\vee(F)}(\tau)$.

Iwahori-Hecke alg.

Let $I(\tau) := \left(\mathrm{Ind}_{B^\vee(F)}^{G^\vee(F)}(\tau)\right)^I \circlearrowleft \mathbb{C}_c[I \backslash G^\vee(F) / I]$

Two bases in $I(\tau)$: ① $\{\varphi_w \mid w \in W\}$, $\varphi_w = 1_{B^\vee(F) w I}$.

② $\{f_w \mid w \in W\}$. Casselman basis (defined using the intertwiners; eigenbasis for the lattice part of the Iwahori-Hecke alg).

define matrix coefficients $m_{u,w}$ by

$$\sum_{w \geq u} \varphi_w = \sum m_{u,w} \cdot f_w$$

Gindikin-Karpelevich formula: $m_{id,w} = \prod_{\substack{\alpha > 0 \\ s_\alpha w < w}} \frac{1 - q^{-1} e^{\alpha}(\tau)}{1 - e^{\alpha}(\tau)}$.

Let $w_0 \in W$ be the longest element.

Conj (Bump-Nakasuji)

Assume the Dynkin diagram of G is simply-laced.

1). For any $u \le w \in W$, the Kazhdan-Lusztig poly. $P_{w_0 w^{-1}, w_0 u^{-1}}(q) = 1$

$$\Longleftrightarrow \quad m_{u,w} = \prod_{\substack{\alpha > 0 \\ u \le s_\alpha w < w}} \frac{1 - q^{-1} e^{\alpha}(\tau)}{1 - e^{\alpha}(\tau)}.$$

2). $\prod_{\substack{\alpha > 0 \\ u \le s_\alpha w < w}} (1 - e^{\alpha})$. $m_{u,w}$ has no pole on $T(\mathbb{C})$.

Remark (Naruse). Remove the condition: "G is simply-laced"

The opposite Schubert variety $Y(u) := \overline{B^- u B / B}$ is smooth at the point $wB \in Y(u)$. $\Longleftrightarrow m_{u,w} = \prod_{\substack{\alpha > 0 \\ u \le s_\alpha w < w}} \frac{1 - q^{-1} e^{\alpha}(\tau)}{1 - e^{\alpha}(\tau)}$.

Thm (Aluffi - Mihalcea - Schürmann - S, 19).

The conjectures hold.

idea: ① G simply laced,

$P_{u,w}(q)=1 \iff X(w)$ is smooth at $uB \in G/B$.

② $\exists!$ affine Hecke algebra module isomorphism

$$K_T(X) \otimes_{K_T(pt)} \mathbb{C}_I \xrightarrow{\sim} I(\tau).$$

s.t. $\{\text{dual basis of } MC_y(X(w)^\circ)\} \longleftrightarrow \{\varphi_w\}$

$\{\text{fixed point basis}\} \longleftrightarrow \{f_w\}$

Iwahori-Whittaker functions.

σ – an unramified principal character of $N^\vee(F)$

Whittaker functional:

$$L : \mathrm{Ind}_{B^\vee(F)}^{G^\vee(F)} \tau \longrightarrow \mathbb{C}, \quad \text{s.t.} \quad L(n\phi) = \sigma(n)\cdot L(\phi), \quad n \in N.$$

For any $f \in \mathrm{Ind}_{B^\vee(F)}^{G^\vee(F)} \tau$,

$$\text{define} \quad W_\tau(f) : G^\vee(F) \longrightarrow \mathbb{C}$$
$$g \longmapsto L(g \cdot f)$$

Spherical Whittaker function. $W_z\left(\sum_{\omega} \varphi_\omega\right) : G^\vee(F) \to \mathbb{C}$.

Thm: (Casselman-Shalika formula.)

μ dominant coweight of $G^\vee$, ϖ - uniformizer of $\mathcal{O}_F$

$$W_{\tau^{-1}}\left(\sum_{\omega} \varphi_\omega\right)(\varpi^{-\mu}) = (*) \prod_{\alpha>0} (1-q^{-1}e^{-\alpha}(\tau)) \cdot \chi_\mu(\tau).$$

char. of irr. highest weight μ rep. of G.

<u>Remark</u>: $\chi_\mu = \chi(G/B^-, G\times_{B^-}\mathbb{C}_\mu) := \sum (-1)^i \mathrm{ch}_T H^i(G/B^-, \mathcal{L}_\mu) \in K_T(pt)$

$G\times_{B^-}\mathbb{C}_\mu =: \mathcal{L}_\mu$

equivariant Euler character.

Iwahori-Whittaker functions:

$$\mathcal{W}_\tau(\varphi_w) : G^\vee(F) \to \mathbb{C}.$$

Thm: (Mihalcea-S, 19). μ dominant coweight of $G^\vee$,

Segre-type class.

$$\mathcal{W}_{\tau^{-1}}(\varphi_w)(\varpi^{-\mu}) \overset{(*)}{=} \prod_{\alpha>0} (1+y e^{-\alpha}(\tau)) \cdot \chi\left(G/B^-, \mathcal{L}_\mu \otimes \frac{MC_y([B^- w B^- \hookrightarrow G/B^-])}{\lambda_y(G/B^-)} \right)$$

Remark: proof uses work of Brubaker-Bump-Licata.

More generally, Brubaker-Buciumas-Bump-Gustafsson. (relation to colored vertex models).

3). Lenart-Zainoulline-Zhong conjecture

hyperbolic formal group law $F_t(x,y) = \dfrac{x+y-xy}{1-(t+t^{-1})^2 xy}$.

$h_T(G/B)$ = oriented cohomology of G/B w.r.t. F_t;

The Konstant-Kumar Hecke alg. acts on $h_T(G/B)$.

The Schubert class $[X(w)]$ is not well-defined if $X(w)$ is not smooth.

Lenart, Zainoulline, and Zhong defined some Kazhdan-Lusztig class

class $\quad KL_w := C_w \cdot ([X(id)])$.

$\uparrow$

Canonical basis in the Hecke algebra

Conj (Lenart-Zainoulline-Zhang)

If $X(w)$ is smooth, then KL_w coincides with the fundamental class $[X(w)]$.

Thm (Lenart-S-Zainoulline-Zhang, 20)

The conjecture holds.

<u>idea</u>: reduce it to the multiplicative case (equiv. K-theory), and use motivic chern classes.

Thank you!