Half-Stationary Vertex Models and Fusion

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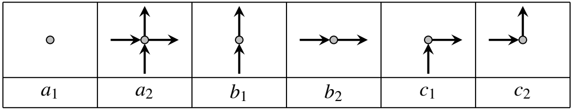
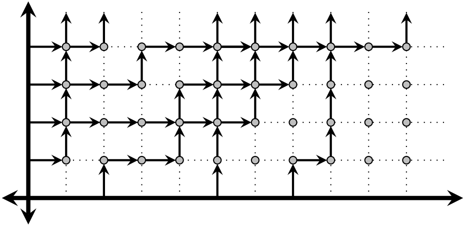
Local Configurations

- Fix some domain $\Lambda \subseteq \mathbb{Z}^2$
- Assign every $v \in \Lambda$ one of six **arrow configurations**, each with a **weight**



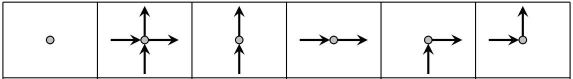
Six-vertex ensemble: Assignment of arrow configuration to each vertex of Λ



- Arrows form up-right directed paths in Λ
- **Boundary conditions** prescribe where paths enter and exit Λ

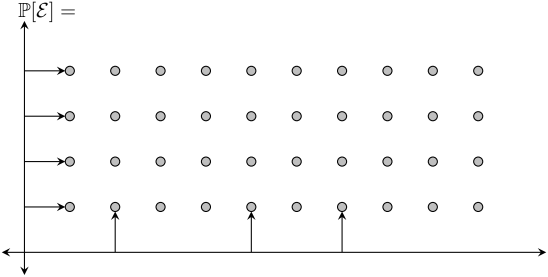
Stochastic Six-Vertex Model

Stochastic six-vertex model (Gwa–Spohn, 1992): Vertex weights are stochastic: Fixing incoming arrows, sum of weights of all configurations is 1



1	1	b_1	b_2	$1 - b_1$	$1 - b_2$

- Enables a local, row by row, Markovian sampling on quadrant $\mathbb{Z}^2_{>0}$
- Markov process on $\{0, 1\}^{\mathbb{Z}_{>0}}$, with y-axis indexing time



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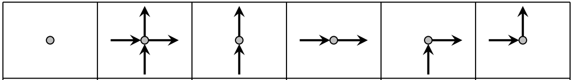

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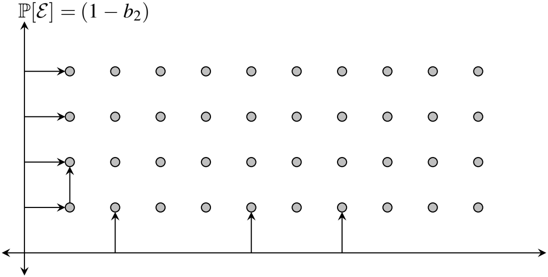

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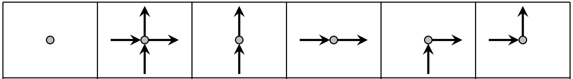

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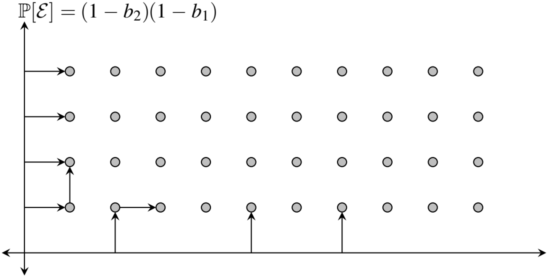

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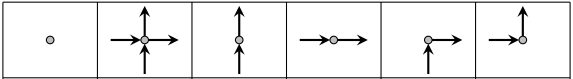

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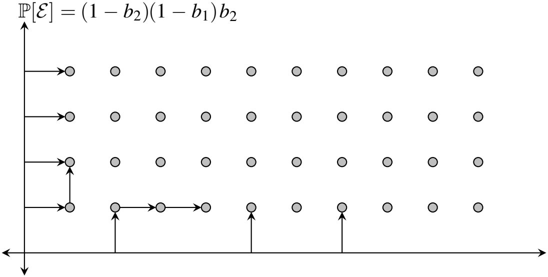

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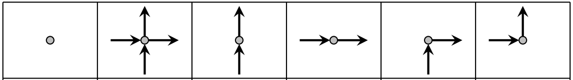

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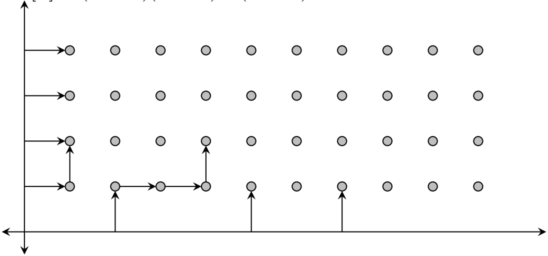

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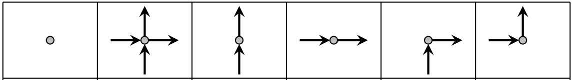

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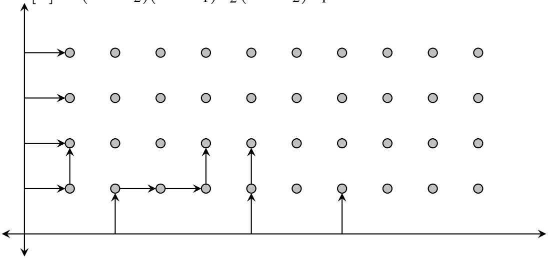

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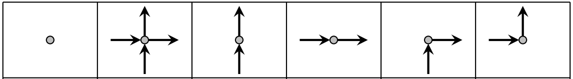

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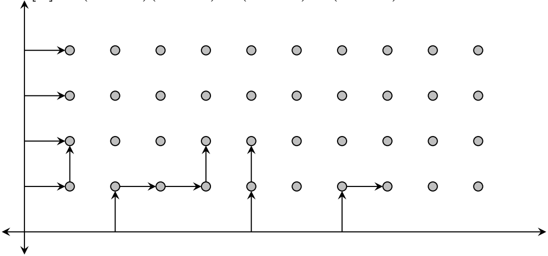

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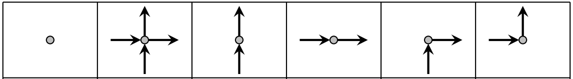

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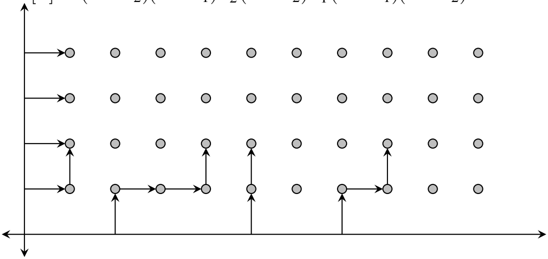

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	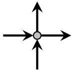		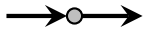		
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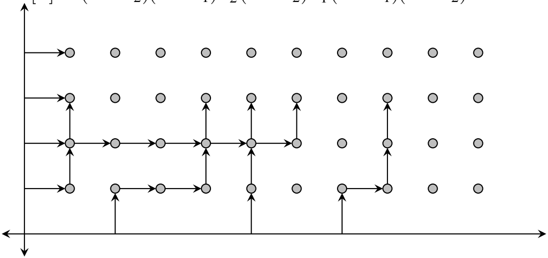

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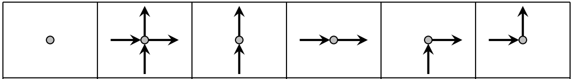

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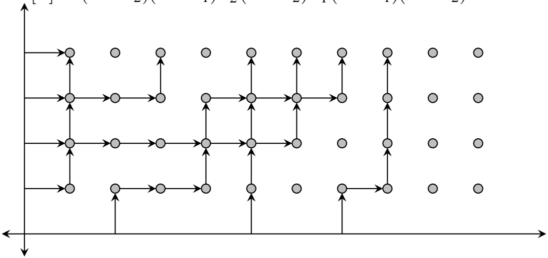

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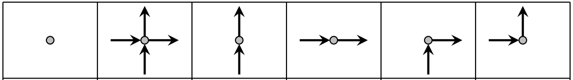

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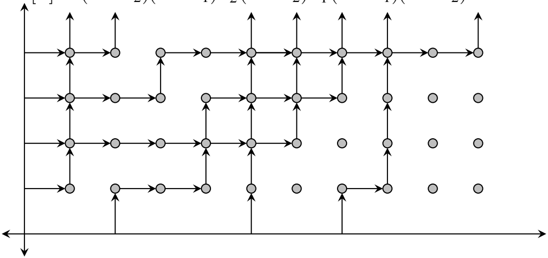
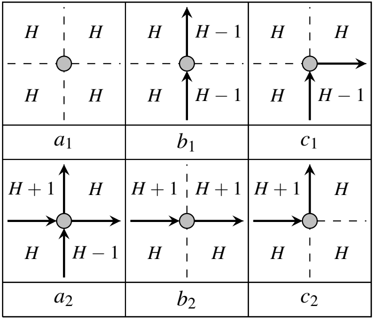
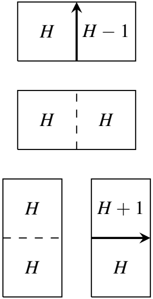


Height Functions

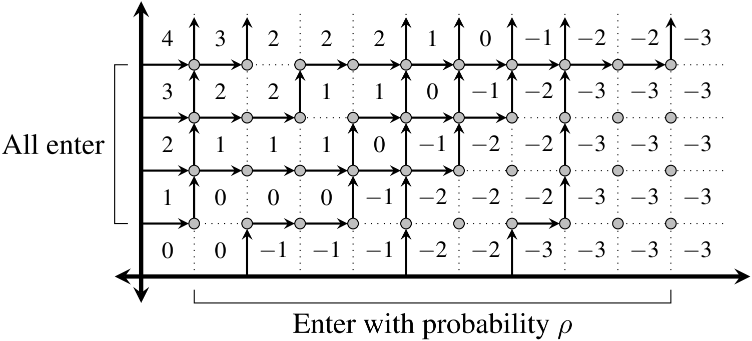




- Assign an integer to each face of the domain, satisfying the above local constraints around every vertex
- This produces a **height function** H on (the dual of) Λ
- Can view $H(u)$ as counting how many paths exist to the right of u
- Bijection between six-vertex ensembles and height function (up to shift)
  - We typically normalize $H(0,0) = 0$

Step-Bernoulli Boundary Data

- ρ **Step-Bernoulli boundary data**
  - Paths enter through all sites of y-axis
  - Paths enter through each site of x-axis independent with probability ρ
- **Step** (partial domain wall) boundary data: $\rho = 0$
  - Paths enter through all sites of y-axis and through no sites of x-axis



Phase Transition

Run stochastic six-vertex model ($b_1 < b_2 < 1$) under ρ step-Bernoulli boundary data

- $H_t(x)$: Height function at $(x, t) \in \mathbb{Z}^2_{>0}$ (height after running model for time t)

Question

For fixed $\xi > 0$, how does $H_T(\xi T)$ behave, as T tends to ∞?

Theorem (A.–Borodin, 2016)

The below limits hold for explicit $\mathcal{H}_\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}$; $\theta_\rho > 0$; $C_\xi, D_\rho, E_{\xi,\rho} \geq 0$.

1. *If $\xi < \theta_\rho$, then $\lim_{T\to\infty} \mathbb{P}\big[H_T(\xi T) \leq T\mathcal{H}_\rho(\xi) + C_\xi T^{1/3} s\big] = F_{GUE}(s)$.*
2. *If $\xi = \theta_\rho$, then $\lim_{T\to\infty} \mathbb{P}\big[H_T(\xi T) \leq T\mathcal{H}_\rho(\xi) + D_\rho T^{1/3} s\big] = F_{GOE}(s)^2$.*
3. *If $\xi > \theta_\rho$, then $\lim_{T\to\infty} \mathbb{P}\big[H_T(\xi T) \leq T\mathcal{H}_\rho(\xi) + E_{\xi,\rho} T^{1/2} s\big] = G(s)$.*

- Known as a **BBP** (Baik–Ben Arous–Péché, 2004) **phase transition**

Original proof based on contour integral identities for q-moments of $H_y(x)$

- Borodin–Gorin–Corwin (2014): Limit shape/fluctuations if $\rho = 0$ (no transition appears)
- A. (2019): Limit shape for any boundary data along axes

Description of $\mathcal{H}_\rho(\xi)$ and θ_ρ

We will describe $\mathcal{H}_\rho$ through its negative derivative $\chi_\rho(\xi) = -\mathcal{H}'_\rho(\xi)$

- Prescribes **local vertical arrow density** near $(\xi T, T)$
  - If there is a vertical arrow exiting (x, t), then $H_t(x+1) - H_t(x) = -1$
  - If there is no vertical arrow exiting (x, t), then $H_t(x+1) - H_t(x) = 0$
- We have $\mathcal{H}_\rho(\xi) = 1 - \int_0^\xi \chi_\rho(\zeta) d\zeta$

Setting $\kappa = \frac{1-b_1}{1-b_2} > 1$, define

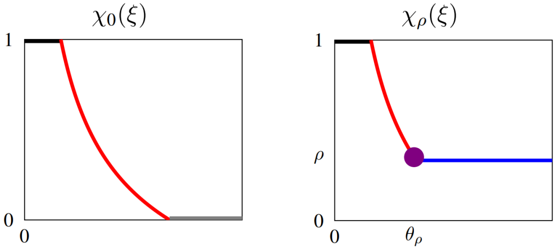
$$\chi_\rho(\xi) = \max\left\{\chi_0(\xi), \rho\right\};$$
$$\chi_0(\xi) = \frac{1}{\kappa - 1}\left(\sqrt{\kappa\xi^{-1}} - 1\right), \quad \text{if } \kappa^{-1} < \xi < \kappa$$

- Also set $\chi_0(\xi) = 1$ if $\xi \le \kappa^{-1}$ and $\chi_0(\xi) = 0$ if $\xi \ge \kappa$
- Then $\chi_0(\xi)$ denotes local density profile for model run under step boundary data
- The profile $\chi_0(\xi)$ is decreasing from 1 to 0 on $[\kappa^{-1}, \kappa]$

Define θ_ρ to be such that $\chi_0(\theta_\rho) = \rho$

- Location where local density under step boundary data equals ρ

Density Plots



Simulation of model under step boundary data

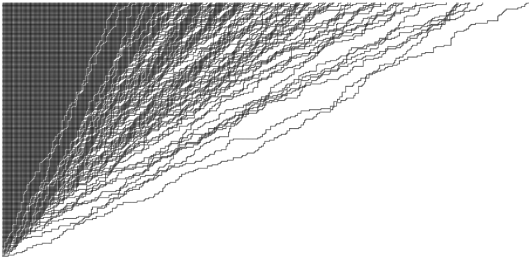
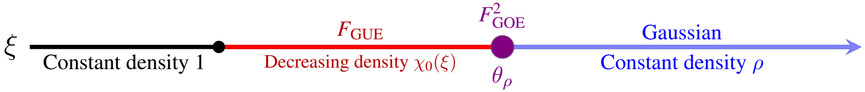


Figure by Leonid Petrov (`https://lpetrov.cc/2015/03/Spin-models`).

Comparison of Density Profiles

ρ Step-Bernoulli boundary data



Step boundary data ($\rho = 0$)

Seems as if one can obtain (most of) step-Bernoulli profile from step profile

- Run stochastic six-vertex model with step boundary data
- Use this step profile to approximate step-Bernoulli profile
  - Left of θ_ρ: Copy step profile
  - Right of θ_ρ: Place arrows with probability ρ, ignoring step profile
    - Also in fact matches Gaussian variance $E^2_{\xi,\rho} = (\xi - \theta)\rho(1 - \rho)$

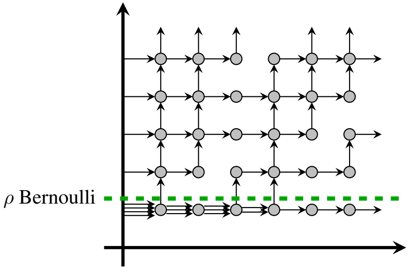
Goal: Explain how this can be heuristically seen using Yang–Baxter equation

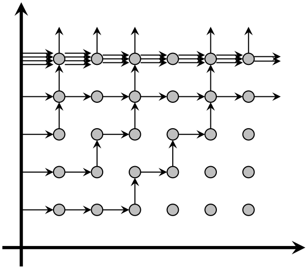
- There are also other probabilistic/analytic interpretations of this phenomenon, but seems most direct through the Yang–Baxter equation

Outline

Goal: Explain how to see transition using Yang–Baxter equation

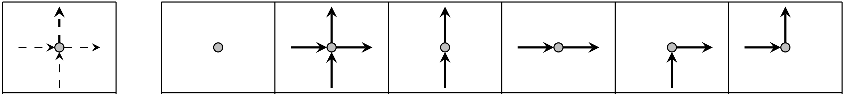
1. Add row operator at bottom of six-vertex model with step boundary data
   - Comes from **fusion** of fundamental solution to Yang–Baxter equation
2. Match the row operator with Bernoulli profile
   - Proceeds through **analytic continuation** of fused weights
3. Use Yang–Baxter equation to commute bottom row operator to the top
   - **Probabilistically interpret** this operator as copying/ignoring profile





Reparameterization and States

- Fix $q \in \mathbb{C}$
- For a **spectral parameter** $u \in \mathbb{C}$ and reparameterize weights as follows



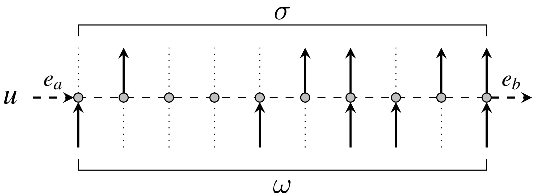
$R_u(i_1, j_1; i_2, j_2)$	1	1	$\frac{q(1-u)}{1-qu}$	$\frac{1-u}{1-qu}$	$\frac{(1-q)}{1-qu}$	$\frac{u(1-q)}{1-qu}$
$(i_1, j_1; i_2, j_2)$	$(0,0;0,0)$	$(1,1;1,1)$	$(1,0;1,0)$	$(0,1;0,1)$	$(1,0;0,1)$	$(0,1;1,0)$

- Define ***R*-matrix** $R(u) = \left[R_u(i_1, j_1; i_2, j_2)\right]$, which is 4×4
- Previous parameterization: Set $q = \frac{b_1}{b_2} < 1$ and $u = \kappa = \frac{1-b_1}{1-b_2}$
- Define $\mathbb{V}_M = V_1 \otimes V_2 \otimes \cdots V_M$, where each V_i is spanned by $\{e_0, e_1\}$
- Interpret basis elements $e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_M}$ as **states** of vertical arrows on level

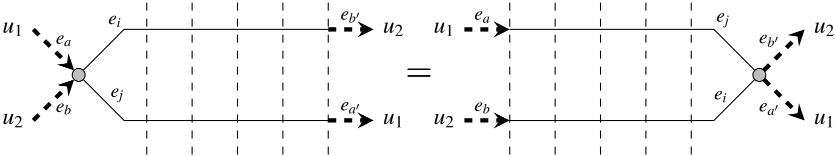
$$e_1 \otimes e_0 \otimes e_0 \otimes e_0 \otimes e_1 \otimes e_0 \otimes e_1 \otimes e_1 \otimes e_0 \otimes e_1 \in \mathbb{V}_{10}$$

Define **transfer matrix** $T_{ab}(u) : \mathbb{V}_M \to \mathbb{V}_M$ through a row partition function

$$\langle \sigma \big| T_{ab}(u) \big| \omega \rangle \quad = $$



Satisfies the **Yang–Baxter equation**



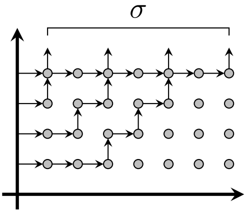
$$\sum_{i,j} R_{u_1/u_2}(b,a;i,j) T_{ib'}(u_2) T_{ja'}(u_1) \qquad \sum_{i,j} T_{aj}(u_1) T_{bi}(u_2) R_{u_1/u_2}(i,j;b',a')$$

Implies **commutation relation** $T_{10}(u_2)T_{10}(u_1) = T_{10}(u_1)T_{10}(u_2)$

Step Profile and Operators

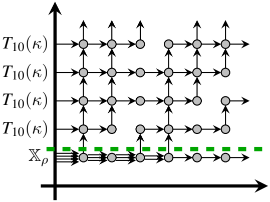
Step boundary data: Probability to see state σ on level N is $\langle \sigma | T_{10}(\kappa)^N | \varnothing \rangle$

$\mathbb{P}[\mathbf{s}_N = \sigma] \quad =$



$= \quad \langle \sigma | T_{10}(\kappa)^N | \varnothing \rangle$

- To obtain ρ step-Bernoulli boundary data, insert an operator $\mathbb{X} = \mathbb{X}_\rho$ before $T_{10}(\kappa)^N$ that "injects" particles into the system
  - Try taking $\mathbb{X} = T_{10}(u_1)T_{10}(u_2)\cdots T_{10}(u_L)$ for some $u_1, u_2, \ldots, u_L$
  - Yang–Baxter equation ensures commutation relation $T_{10}(\kappa)^N \mathbb{X}_\rho = \mathbb{X}_\rho T_{10}(\kappa)^N$



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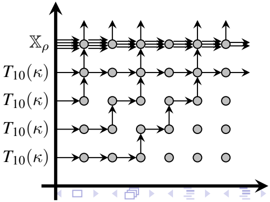


Fusion

- Try taking $\mathbb{X} = T_{10}(u_1)T_{10}(u_2)\cdots T_{10}(u_L)$ for some $u_1, u_2, \ldots, u_L$

Issue: Would like to explicitly evaluate action of $\mathbb{X}$

- Given by a L-row partition function (highly intricate for arbitrary $u_1, u_2, \ldots, u_L$)

Kulish–Reshetikhin–Sklyanin (1981): **Fusion**

- Suppose that $R(\gamma)$ is a projection for some $\gamma \in \mathbb{C}$
- Then, $T_{10}(u)T_{10}(\gamma u)\cdots T_{10}(\gamma^{L-1}u)$ simplifies considerably
  - The L-row partition functions becomes a single-row one under certain new weights
  - These new **fused weights** satisfy the Yang–Baxter equation

Holds for $\gamma = q$, as then $R(q) = \begin{bmatrix} 1 & & & \\ & \frac{q}{1+q} & \frac{1}{1+q} & \\ & \frac{q}{1+q} & \frac{1}{1+q} & \\ & & & 1 \end{bmatrix}$

- Also holds if $\gamma = q^{-1}$

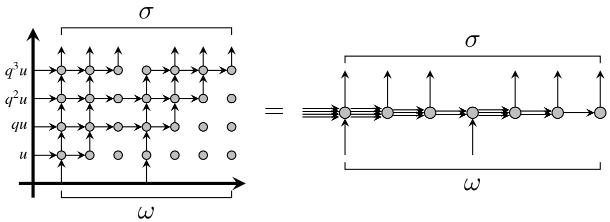
Set $\mathbb{B}(u) = \mathbb{B}(u; q^{-L}) = T_{10}(u)T_{10}(qu)\cdots T_{10}(q^{L-1}u)$

Fused Vertices

The L-row partition function $\mathbb{B}(u)$ becomes single row one under new fused weights

- Concatenate the L rows to form one

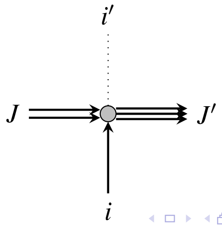
$$\langle \sigma | \mathbb{B}(u) | \omega \rangle =$$



Fused vertices are of the following form (below, $(i, J; i', J') = (1, 2; 3, 0)$)

- Allow at most L arrows along horizontal edges and one along vertical edges

$$W_u(i, J; i'; J' | q^L) =$$

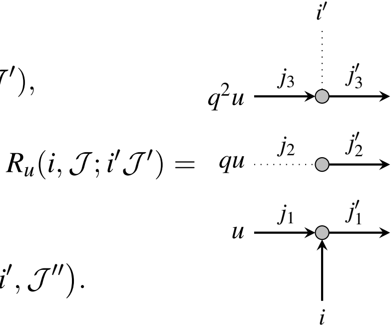


Evaluating the Fused Weights

Define for indices $i, i' \in \{0,1\}$ and sequences $\mathcal{J} = (j_1, j_2, \ldots, j_L) \in \{0,1\}^L$. $\mathcal{J}' = (j'_1, j'_2, \ldots, j'_L) \in \{0,1\}^L$, **column weights** $R_u(i, \mathcal{J}; i', \mathcal{J}')$.

Set

$$\mathcal{W}_u(i, J; i', \mathcal{J}') = \sum_{|\mathcal{J}|=J} q^{\mathrm{inv}(\mathcal{J})} R_u(i, \mathcal{J}; i', \mathcal{J}'),$$



where inv counts inversions.

Then $\mathcal{W}$ is q-**exchangeable**:

$$\mathcal{W}_u(i, J; i', \mathcal{J}') = q^{\mathrm{inv}(\mathcal{J}') - \mathrm{inv}(\mathcal{J}'')} \mathcal{W}_u\big(i, J; i', \mathcal{J}''\big).$$

Define the fused weight

$$W_u(i, J; i', J' | q^L) = \mathcal{W}_u(i, J; i', \mathcal{J}'),$$

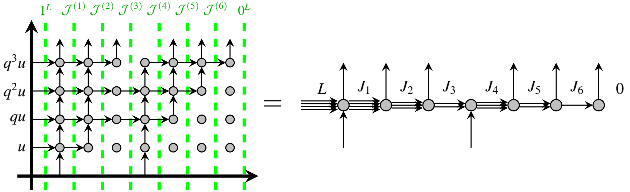
where $\mathcal{J}' = 0^{L-J'} 1^{J'}$ (so $\mathrm{inv}(\mathcal{J}') = 0$).

Concatenation

Recall

$$\mathcal{W}_u(i,J;i',\mathcal{J}') = \sum_{|\mathcal{J}|=J} q^{\mathrm{inv}(\mathcal{J})} R_u(i,\mathcal{J};i',\mathcal{J}'); \qquad W_u(i,J;i',J'|q^L) = \mathcal{W}_u(i,J;i',0^{L-J'}1^{J'});$$

$$\mathcal{W}_u(i,J;i',\mathcal{J}') = q^{\mathrm{inv}(\mathcal{J}')-\mathrm{inv}(\mathcal{J}'')}\mathcal{W}_u\big(i,J;i',\mathcal{J}''\big)$$



The left side equals

$$\sum_{\mathcal{J}^{(i)}} R_u(i_1,1^L;i_1',\mathcal{J}^{(1)})R_u(i_2,\mathcal{J}^{(1)};i_2',\mathcal{J}^{(2)})\cdots = \sum_{\mathcal{J}^{(i)}} q^{\mathrm{inv}(\mathcal{J}^{(1)})}\mathcal{W}_u(i_1,L;i_1',0^{J_1}1^{L-J_1})R_u(i_2,\mathcal{J}^{(1)};i_2',\mathcal{J}^{(2)})\cdots$$

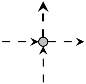
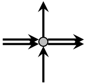
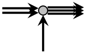
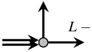
$$= W_u(i_1,L;j_1',J_1)\sum_{\mathcal{J}^{(i)}} q^{\mathrm{inv}(\mathcal{J}^{(1)})}R_u(i_2,\mathcal{J}^{(1)};i_2',\mathcal{J}^{(2)})\cdots = W_u(i_1,L;j_1',J_1)\sum_{\mathcal{J}^{(i)}}\mathcal{W}_u(i_2,J_1;i_2',\mathcal{J}^{(2)})\cdots$$

$$= W_u(i_1,L;j_1',J_1)\sum_{\mathcal{J}^{(i)}} q^{\mathrm{inv}(\mathcal{J}^{(2)})}\mathcal{W}_u(i_2,J_1;i_2',0^{L-J_2}1^{J_2})\cdots = W_u(i_1,L;j_1',J_1)W_u(i_2,J_1;i_2',J_2)\cdots,$$

which is the right side

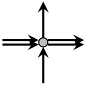
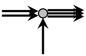
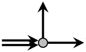
Fused Weights

Above framework enables $W_u(i,J;i',J'|q^L)$ to be solved recursively in L

	$L-d$  $L-d$	$L-d$  $L-d$	$L-d$  $L-d+1$	$L-d$  $L-d-1$
$W_u(i,J;i',J')$	$\frac{1-q^{d+1-L}u}{1-qu}$	$\frac{q^d-qu}{1-qu}$	$\frac{1-q^d}{1-qu}$	$\frac{qu(q^{d-L}-1)}{1-qu}$
$(i,J;i;,J')$	$(0,L-d;0,L-d)$	$(1,L-d;1,L-d)$	$(1,L-d;0,L-d+1)$	$(0,L-d;L-d-1,1)$

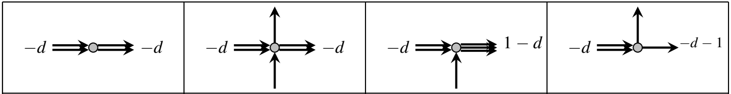
Weights are rational in $\alpha = q^{-L}$

- Replace all arrow configurations $(i,J;i',J')$ with $(i,J-L;i',J'-L)$
- Tracks **deficit** $J-L=-d$, which cannot be less than $-L$ or more than 0
  - These constraints are guaranteed by factors $q^{d-L}-1=\alpha q^d-1$ and $1-q^d$

$-d$  $-d$	$-d$  $-d$	$-d$  $1-d$	$-d$  $-d-1$
$\frac{1-\alpha q^{d+1}u}{1-qu}$	$\frac{q^d-qu}{1-qu}$	$\frac{1-q^d}{1-qu}$	$\frac{qu(\alpha q^d-1)}{1-qu}$
$(0,-d;0,-d)$	$(1,-d;1,-d)$	$(1,-d;0,1-d)$	$(0,-d;-d-1,1)$

Analytic Continuation

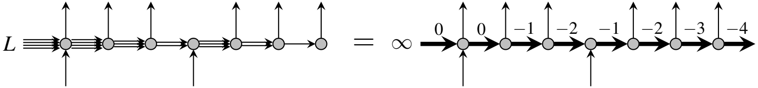
Would like to analytically continue in $\alpha = q^{-J}$



$\frac{1-\alpha q^{d+1}u}{1-qu}$	$\frac{q^d - qu}{1-qu}$	$\frac{1-q^d}{1-qu}$	$\frac{qu(\alpha q^d - 1)}{1-qu}$

Issue: We must have $L \in \mathbb{Z}_{\geq 0}$, since arrows enter through the fused row

- Factors $\alpha q^d - 1$ and $1 - q^d$ enable us to let infinitely many paths exist in fused row
- Then, the number of arrows $L - d$ at any point in the fused row no longer well-defined
  - Deficit d is still well-defined (arrows entered row subtracted from arrows exited the row)



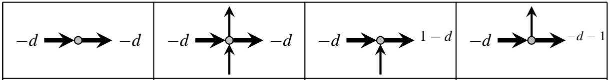
Right side induces row operator on $\mathbb{V}_M$, denoted by $\mathbb{B}(u;\alpha)$, for any $\alpha \in \mathbb{C}$

- **Analytic continuation** of $\mathbb{B}(u; q^{-L})$
- **Preserves commutation relation** $T_{01}(\kappa)^N \mathbb{B}(u;\kappa) = \mathbb{B}(u;\alpha) T_{01}(\kappa)^N$

The Operator $\mathbb{X}_\rho$ and ρ Step-Bernoulli Boundary Data

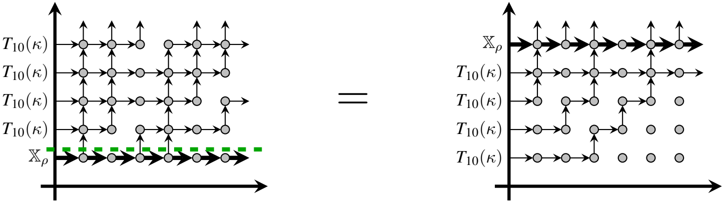
Set $\mathbb{X}_\rho = \mathbb{B}(u;\alpha)$, where $\alpha = 0$ and $\frac{qu}{qu-1} = \rho$

- Equivalently, $L = -\infty$ (since $q < 1$) and $u = \frac{\rho}{q(\rho-1)}$



$1-\rho$	$q^d(1-\rho)+\rho$	$(1-q^d)(1-\rho)$	ρ

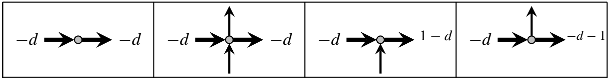
Action on e_0: $\mathbb{X}_\rho$ outputs vertical arrow with probability ρ (irrelevant of d)



Left side produces ρ step-Bernoulli boundary data

- Similar framework yields (half-)stationary data for colored vertex models
  - Martin (2018): Multi-line queue diagrams
  - Corteel–Mandelshtam–Williams (2018): Weighted tableaux

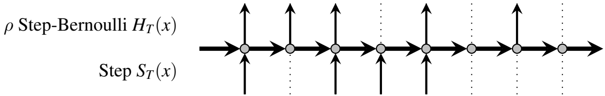
Action of $\mathbb{X}_\rho$ on Step Profile



$1-\rho$	$q^d(1-\rho)+\rho$	$(1-q^d)(1-\rho)$	ρ

By commutation, $H_T(x)$ under ρ step-Bernoulli can be sampled as follows

- Run T steps of stochastic six-vertex model under step boundary data
- Apply the operator $\mathbb{X}_\rho$ to the output



Let $S_T(x)$ denote height function of model under step boundary data

- Denoting the deficit at site x by $d(x)$, we have $H_T(x) = S_T(x) - d(x)$
- **Copying phase**: If $d \ll T^{1/3}$ is small

For $d \gg 1$, we have $q^d \approx 0$, so $q^d(1-\rho)+\rho \approx \rho$ and $(1-q^d)(1-\rho) \approx 1-\rho$

- Vertex weights $W(i, -d; i', -d')$ are approximately independent of i
- **Ignoring phase**: If $d \gg 1$ is large

Deficit Behavior and Local Densities

$-d$ → $-d$	$-d$ → $-d$	$-d$ → $1-d$	$-d$ → $-d-1$
$1-\rho$	$q^d(1-\rho)+\rho$	$(1-q^d)(1-\rho)$	ρ

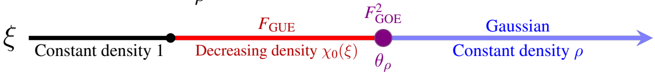
- Copying phase for small d and ignoring phase for large d
- We start at $x=0$ with $d=0$ (copying phase)

Analyze circumstances under which copying phase can turn to ignoring phase

- Recall "local vertical arrow density" $\chi_0 = \chi_0(\xi)$ in step profile near $(\xi T, T)$
- Increase d: Probability ρ if no vertical arrow in step profile
  - Near $(\xi T, T)$, happens with approximate proportion $(1-\chi_0)\rho$
- Decrease d: Probability $(1-q^d)(1-\rho) \approx 1-\rho$ above vertical arrow
  - Near $(\xi T, T)$ happens with approximate proportion $(1-\rho)\chi_0$

Increase d if $(1-\chi_0)\rho > (1-\rho)\chi_0$ (namely, $\chi_0 > \rho$); decrease d if $\chi_0 > \rho$

- Transition from small d to large d occurs when $\chi_0(\xi) = \rho$
- This is the definition of θ_ρ



In fact implies height fluctuations converge to Brownian motion to the right of $\theta(\xi)$

- Not entirely transparent how to directly see this from moment identities

Summary

- Phase transition stochastic six-vertex model under ρ step-Bernoulli boundary data
  - For $\xi < \theta_\rho$, fluctuations of $H_T(\xi T)$ are F_{GUE} and of order $T^{1/3}$
  - For $\xi = \theta_\rho$, fluctuations of $H_T(\xi T)$ are F_{GOE}^2 and of order $T^{1/3}$
  - For $\xi > \theta_\rho$, fluctuations of $H_T(\xi T)$ are Gaussian and of order $T^{1/2}$
- Comparison to step asymptotic behavior suggests the following
  - "Copying phase" to the left of θ_ρ
  - "Ignoring phase" to the right of θ_ρ
- The Yang–Baxter equation provides a way of seeing this
  - Insert fused row operator to inject particles in systematic way
  - Analytic continuation of weights to obtain Bernoulli profile
  - Probabilistic interpretation to see phases from new operator
    - Copying phase: Small deficit d
    - Ignoring phase: Large deficit
    - Small (or large) d when local arrow density $\chi_0 > \rho$ (or $\chi_0 < \rho$, respectively)