Integrable Systems Conference Lecture

by

Ben Brubaker
(in collab. w/ Buciumas, Bump, Gustafsson)

Integrable Systems and p-adic Representation Thy.

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In collaboration with: Buciumas, Bump, Gustafsson [3BG]
(primarily focus on 2 recent papers w/ arXiv #'s:
1902.01795, 1906.04140, + one in progress)

See also: "Frozen Pipes" 2007.04310
Solvable lattice models for Grothendieck polys (double β-)
with C. Frechette, A. Hardt, E. Tibor, K. Weber

See also also: Buciumas - Scrimshaw (2020).

Plan for the talk:

- Overview of ubiquity of lattice models
- Brief overview of objects in p-adic repn thy.
- How deep does connection to lattice models go?
- New wrinkles in lattice models arising from our story.

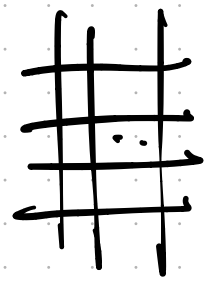
The Big Picture about Lattice Models:

(from my limited point of view)

QUANTUM GROUP MODULE

$\updownarrow$ YBE

SOLVABLE LATTICE MODELS



Partition function + subsequent identities

integrable probability.

gen'd cohomology of flag varieties

p-adic repn theory.

Harish-Chandra's philosophy: Prove theorems for all semisimple gps simultaneously.

Goal: Use lattice models as a bridge to methods / connex. to quantum gps which might hold in great generality.

A Primer on Matrix Coeffs. of p-adic Groups.

Given (π, V) rep'n of $G(F)$: split red. alg. gp / p-adic field F,

$\mathcal{L}$: linear functional from
$V \to \mathbb{C}$

$v_0 \in V$. Then define $\phi(g) := \mathcal{L}(\pi(g) \cdot v_0)$

Example: Whittaker functional on (π, V) : unramified principal series.

$\mathcal{L}$: Whittaker functional is made via integration against character ψ of $\mathcal{U}^-$: opposite unip in $G \supset B = T\mathcal{U}$ over $\mathcal{U}^-$

(π, V) : $\chi : T(F)/T(\mathcal{O}) \to \mathbb{C}^\times$, inflate to B induce to G.

$\mathcal{O}$: ring of ints of F

$\chi_{\underline{z}} : \begin{pmatrix} \varpi^{n_1} & & \\ & \ddots & \\ & & \varpi^{n_r} \end{pmatrix} \longmapsto z_1^{n_1} \cdots z_r^{n_r}$ Call the result $\pi_{\underline{z}}$

(Shintani, Kato, Casselman-Shalika) 80's

$$\mathcal{L}(\pi_{\underline{z}}(t_\lambda) \cdot v_0) = \begin{cases} \prod_{\alpha \in \Phi^+} (1 - q^{-1} z^\alpha)\, s_\lambda(\underline{z}) & \\ 0 & \text{else} \end{cases}$$

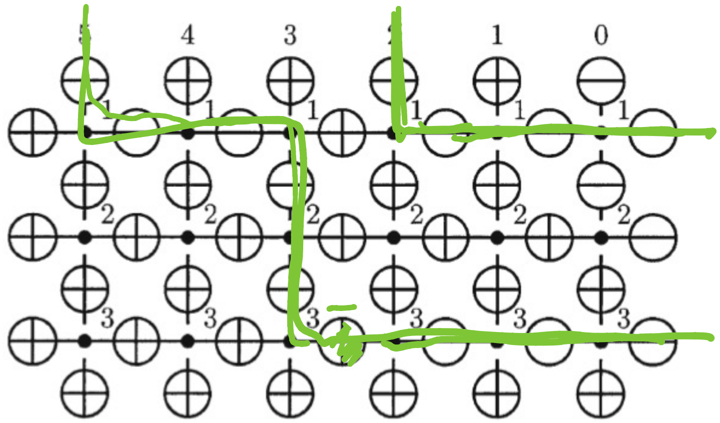
$s_\lambda(\underline{z})$ ← char. of h.w. λ on ${}^L G$

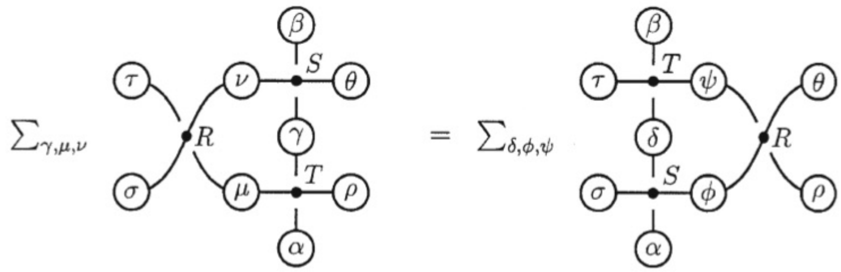
$t_\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_r} \end{pmatrix}$, v_0 : $G(\mathcal{O})$-fixed vector in (π_z, V) "spherical vector"

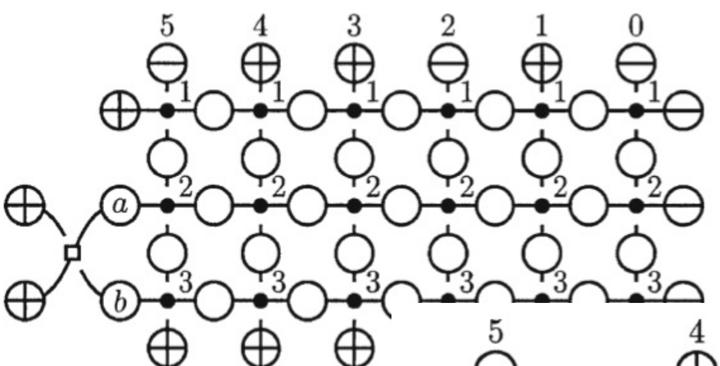
Ancient History - (B.-Bump-Friedberg '09)

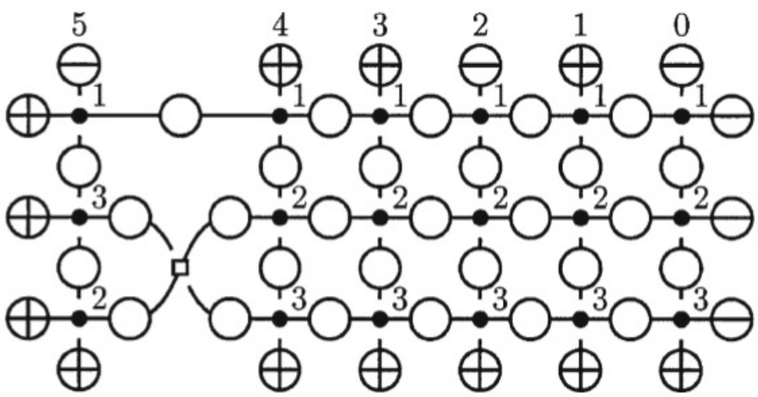
Tokuyama

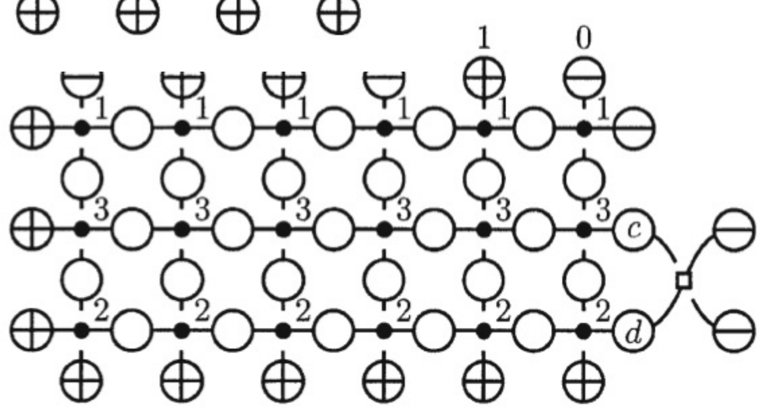
Hamel-King.





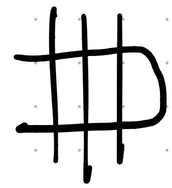






Levers we can pull... (Generalizations)

- change group : other classical gp



- change vector : Pick vector fixed by smaller compact s.g.

  I : Iwahori

  $G(\mathcal{O}) = \dot{\bigcup}_{w \in W} IwI$

- change the functional : integrate against other s.g.'s. $G(\mathcal{O})$-integ. $\rightsquigarrow$ Hall-Littlewood polys.

- introduce covers of gp. $1 \to \underbrace{\mu_n}_{n^{th}\text{ rts of unity}} \to \tilde{G} \to G(F) \to 1.$

- change rep'n.

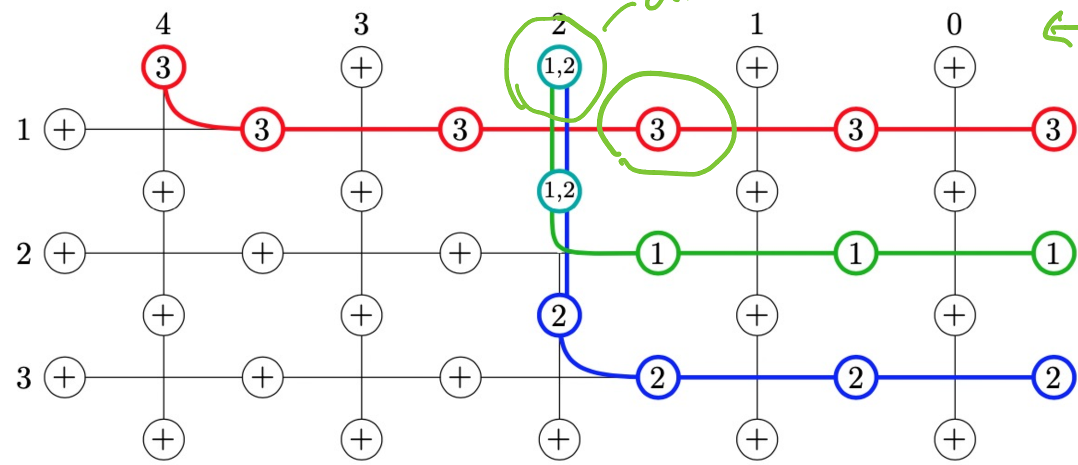
	proto	$GL(r)$	$\widetilde{GL}(r)$
$\boxed{\text{Spherical } G(\mathcal{O})}$	Schur polys	Casselman-Shalika on 6-vertex [BBF]	met. C-S. on $6n^2$ vertex model. [BBB]
colored lattice models. Borodin-Wheeler (2018) ⇝	⇝	⇝	⇝
$\boxed{\text{Iwahori } I}$ [3BG]	Demazure atoms	Iwahori Whittaker fncs.	Metaplectic Iwahori

Iwahori Whittaker Functions + Lattice Models.

distinct.

$\leftarrow \lambda$

boring

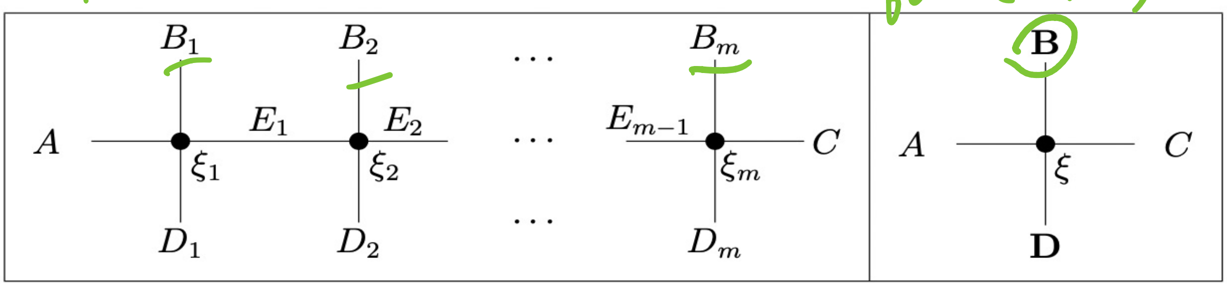


perm. of colors indicate Iwahori fixed vector

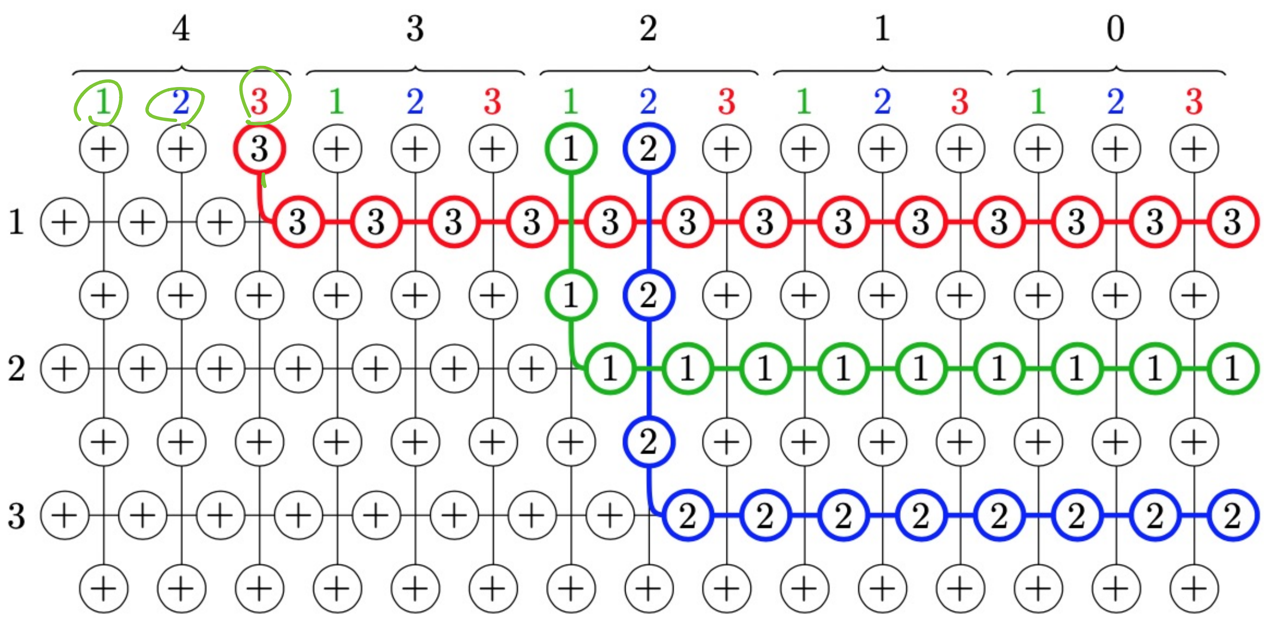
YBE are Demazure like operators.

boring

Horizontal edges: $U_q(\widehat{\mathfrak{sl}}(r|1))$



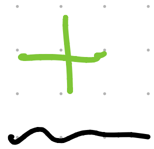
via fusion



A few notes about Iwahori Whittaker lattice models:

- the model is fermionic (no superposition of same particles) but multiple distinct particles may occupy columns.

- we must use combinatorial version of fusion, not tied to a quantum gp module interp., in order to render YBE a finite computation.

- The R-matrix is (a Drinfeld twist of) an r+1-dim'l module for $U_v(\hat{gl}(r|1))$, but in fact quantum gp can vary (can be $U_v(\hat{gl}(r+1))$) according to Boltzmann weight of 

↑ quantum param related

any vertex with <u>same</u> color intersecting itself.

Since vertices like these do not appear in our lattices (according to our choice of boundary) we are free to choose either, but we'll say more about this in a minute.

Comments about rep'n theory in this model:

- Just as in M. Wheeler's talk, Iwahori Whittaker functions are expressed as Demazure-like ops. acting on easily evaluated initial state. ("ground state")

- Initially, we only allowed boundaries corresp. to λ: dominant wt. in $W(\phi)(t_\lambda)$.

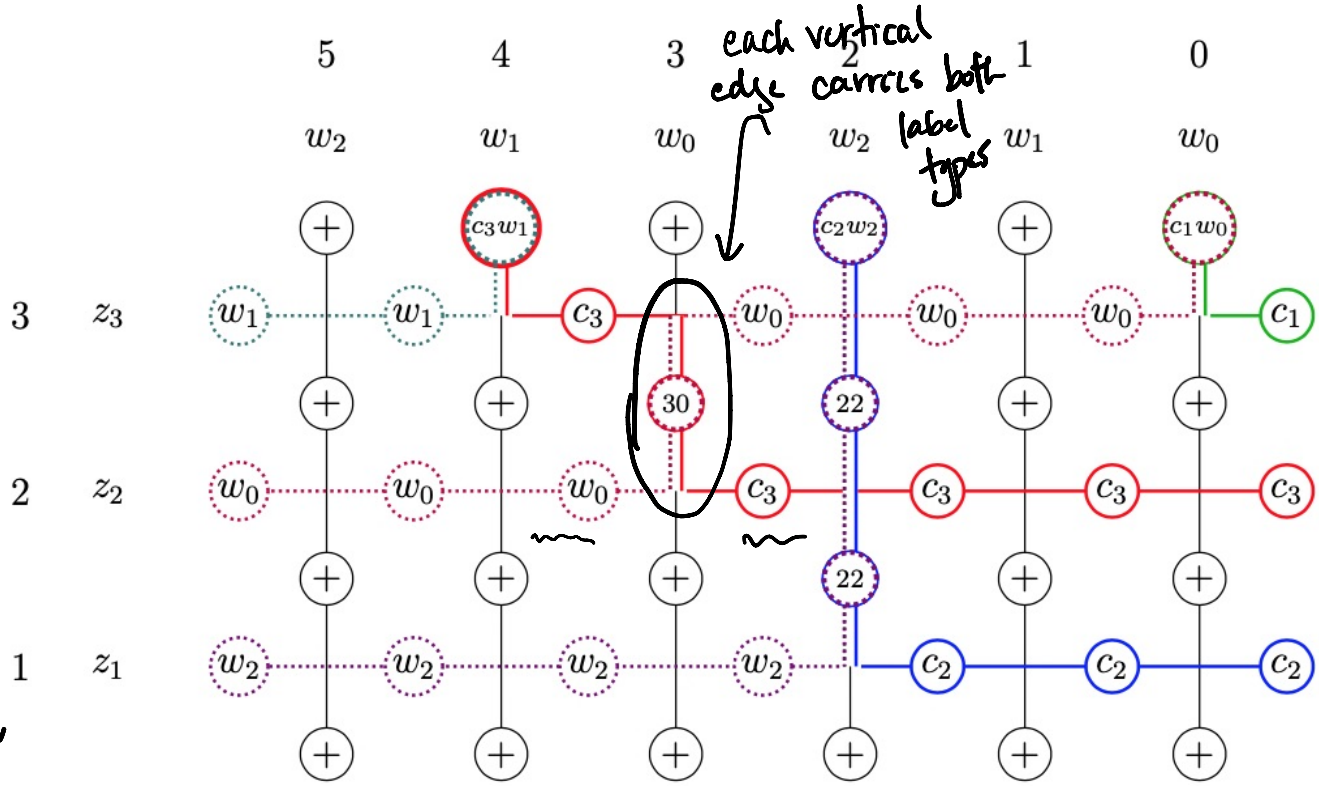
But with Iwahori-fixed vectors, need larger set of values to determine $W(\phi)$. Won't be precise, but we must evaluate at $t_\lambda w$ with $t_\lambda \in T(F)$: torus. where λ is "w-almost-dominant" $w \in W$: Weyl gp
(see paper for def'n)

Pairs (λ, w) are in bijection with compositions and remarkably, using that composition in the lattice boundary, the resulting partition function MATCHES! the Whittaker function value $W(\phi)(t_\lambda w)$.

- Can also evaluate parahoric fixed vectors, and in this model, $+$ vertices are necessary. Reveals that quantum superalgebra is the "right choice" for rep. theory.

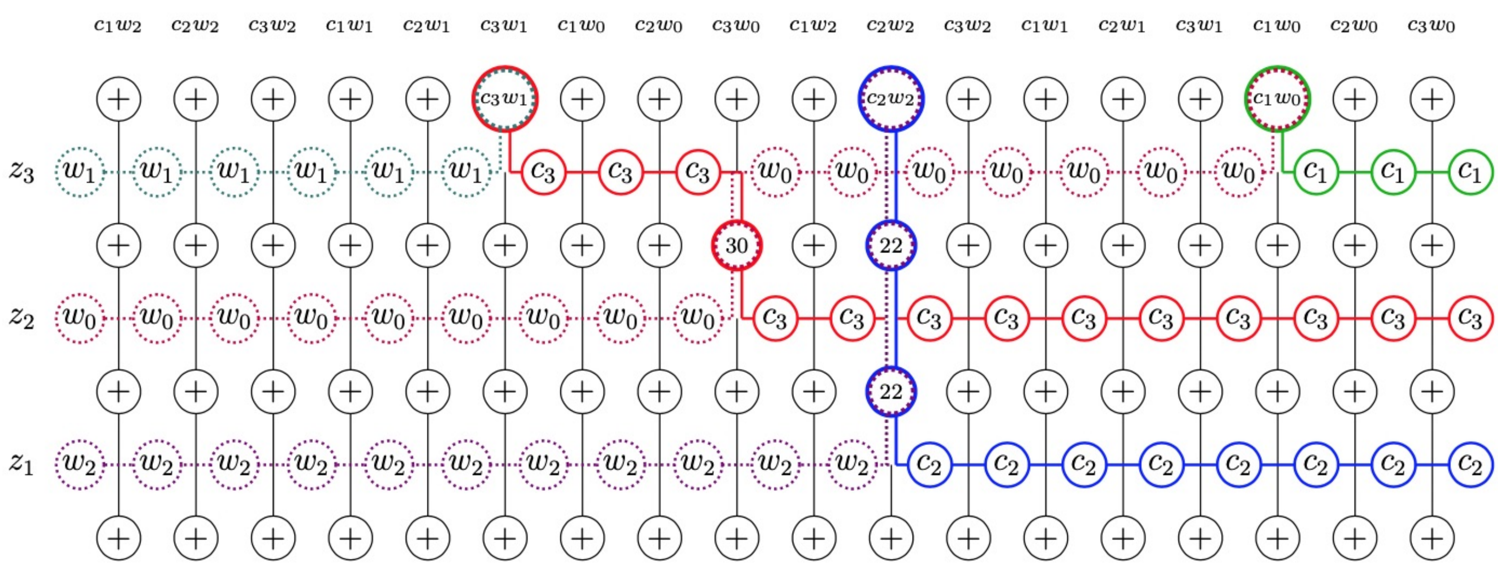
each vertical edge carries both label types

met. wh. functionals

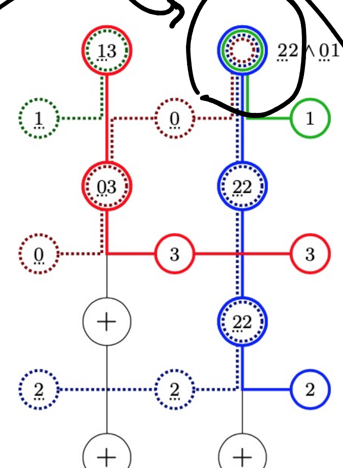


Iwahori

fusion



We can have multiple distinct labels



"fully fused system"

Associated quantum gp: $\mathcal{U}_v(\widehat{gl}(r|n))$

where n: cover degree in

$M_n \to \tilde{G} \to G(F)$.

See Poulain d'Andecy for comb. descr. of fusion.