How coordinate Bethe ansatz works for Inozemtsev model

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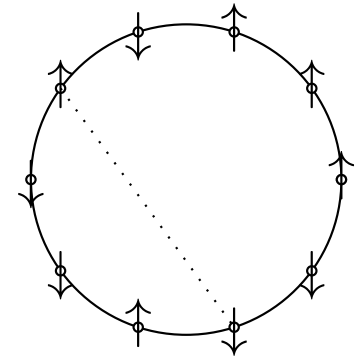
joint work with **Rob Klabbers** (Nordita)

building on [V. I. Inozemtsev, '90–'99]

Exactly solvable long-range spin chains



	nearest-neighbour (contact)	intermediate (elliptic)	long range (trigonometric)
anisotropic (elliptic)	(some version of) Heisenberg XYZ	anisotropic Inozemtsev ??	anisotropic Haldane–Shastry ??
partially isotropic (trig)	(some version of) Heisenberg XXZ	q-deformed Inozemtsev ?	q-deformed Haldane–Shastry
isotropic (rational)	Heisenberg XXX	Inozemtsev	Haldane–Shastry

Inozemtsev's elliptic spin chain
Anatomy

[Inozemtsev '90, '95]

Weierstrass $\wp$
periods $(L, \mathrm{i}\pi/\kappa) \in \mathbb{N} \times \mathrm{i}\,\mathbb{R}_{>0}$

long-range pairwise

spin exchange

$$H = \sum_{i<j}^{L} \overbrace{\mathrm{cst}(\kappa)\big(\wp(i-j) + \mathrm{cst}(L,\kappa)\big)}^{\text{elliptic pair potential}} \frac{\vec{\sigma}_i \cdot \vec{\sigma}_j - 1}{2}$$

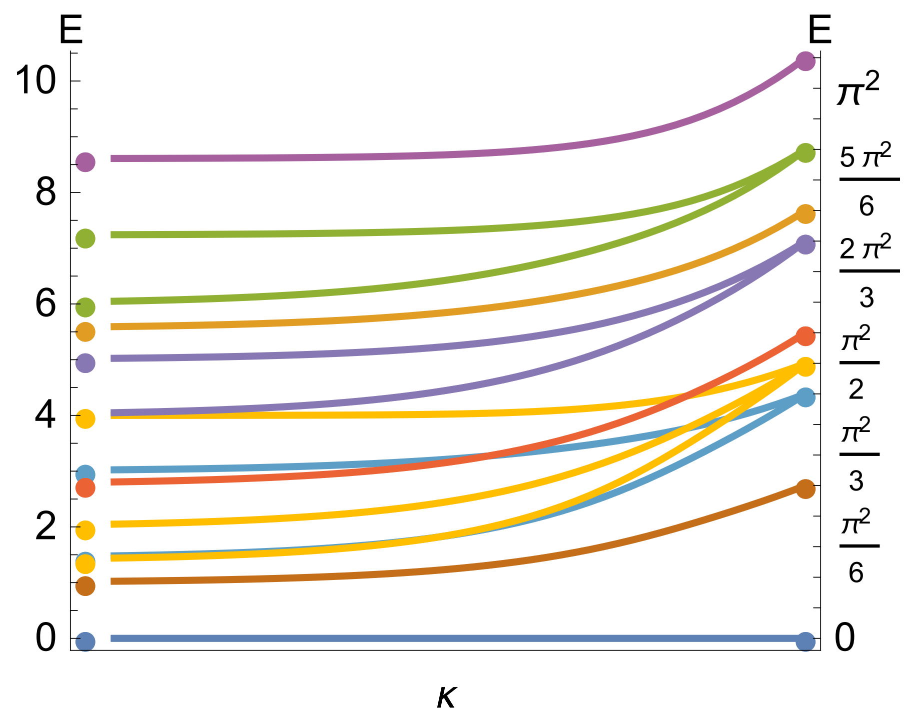
$\kappa \to \infty$: $\delta_{d_L(i-j),1}$ [Heisenberg '28]

$L \to \infty$: $\dfrac{\sinh^2 \kappa}{\sinh^2 \kappa(i-j)}$ [Inozemtsev '92]

$\kappa \to 0$: $\dfrac{(\pi/L)^2}{\sin^2(\pi(i-j)/L)}$ [Haldane '88] [Shastry '88]

Inozemtsev's elliptic spin chain
Exactly solvable interpolation

Heisenberg
- **Exactly solvable** up to solving Bethe-ansatz equations
- **Quantum integrable** $\mathrm{tr}_a\, L_a(u)$



Haldane–Shastry
- **Exactly solvable** in closed form (with Jacks)
- **Quantum integrable** $\mathrm{qdet}_a\, L_a(u;\{d_i\})$
- **Yangian symmetry**

Inozemtsev
- **Exactly solvable** spectrum up to solving BAE
- **Quantum integrable?**

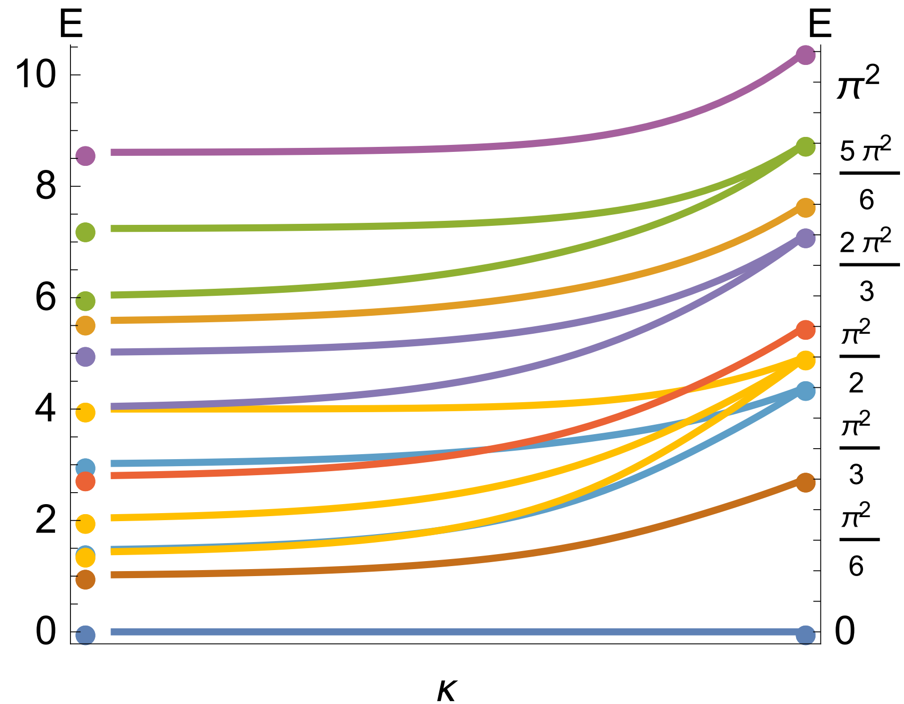
Inozemtsev's elliptic spin chain
Some highlights

- Spectrum for M=2 [Inozemtsev '90]
  using Hermite's solution of **Lamé equation**
- Spectrum in **hyperbolic limit** [Inozemtsev '92]
  via connection to **hyperbolic Calogero–Sutherland** using [Chalykh Veselov '90]
- Spectrum for **general** M [Inozemtsev '95, '99]
  via connection to **elliptic Calogero–Sutherland** using [Felder Varchenko '95]
- Asymptotic **Yangian** symmetry [Ha Haldane '93] [De La Rosa Gomez *et al* '16]
- **Thermodynamic** Bethe ansatz [Dittrich Inozemtsev '97] [Klabbers '16]
- **Proposal** for **higher Hamiltonians** [Inozemtsev '96]
- Guest appearance in **AdS/CFT** [Serban Staudacher '04]

Our goal

$$H = \sum_{i<j}^{L} \mathrm{cst}(\kappa)\big(\wp(i-j) + \mathrm{cst}(L,\kappa)\big)\,\frac{\vec{\sigma}_i \cdot \vec{\sigma}_j - 1}{2}$$

Understand Inozemtsev's solution and its **limits**



Strategy
General considerations

- **Isotropy** ($\mathfrak{sl}_2$-invariance): fix $M = \#\!\downarrow$, focus on highest weight
- Use coordinate basis $\sigma^-_{n_1} \cdots \sigma^-_{n_M} |{\uparrow} \cdots {\uparrow}\rangle$
  (so will have to ensure **cyclicity**)
- Homogeneity (translational invariance) determines $M = 1$
  Dispersion relation reciprocal periods $(2\pi, 2\mathrm{i}\kappa)$ [Inozemtsev '90] [Klabbers JL]

$$\varepsilon(p) = \mathrm{cst}(\kappa) \left(\wp^\vee(p) - \Big(\zeta^\vee(p) - \frac{\zeta^\vee(\pi)}{\pi}\, p\Big)^2 - 2\, \frac{\zeta^\vee(\pi)}{\pi} \right)$$

$\kappa \to \infty$: $4 \sin^2(p/2)$ Heisenberg

$\kappa \to 0$: $p\,(2\pi - p)/2$ Haldane–Shastry

Strategy
Extended coordinate Bethe ansatz

[Inozemtsev '95]
[Klabbers JL]

For general M seek wave functions of the form

$$\Psi_{\boldsymbol{p}}(\boldsymbol{n}) = \sum_{w \in S_M} \widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}_w)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\tilde{\boldsymbol{p}})\cdot \boldsymbol{n}_w}$$

plane wave

depends on **positions**!

Assumptions

- [technical] $\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}$ has **simple poles** at **equal arguments**
- **double quasiperiodicity** $\begin{cases} \widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}+\mathrm{i}\pi\kappa^{-1}\hat{e}_m) = \mathrm{e}^{-\pi\kappa^{-1}\tilde{p}_m}\,\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}) \\ \widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}+L\hat{e}_m) = \mathrm{e}^{\mathrm{i}(L\tilde{p}_m-\varphi_m)}\,\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}) \end{cases}$

Cyclicity of $\Psi_{\boldsymbol{p}}$ requires **Bethe-ansatz equations**

$$L\,p_m = 2\pi I_m + \varphi_m\,, \qquad I_m \in \mathbb{Z}_L\,, \qquad 1 \leq m \leq M$$

Connection with eCS
Result of extended CBA

If $\widetilde{H}|_{\beta=2}\,\widetilde{\Psi}_{\tilde{\boldsymbol{p}}} = \widetilde{E}\,\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}$ for **quantum elliptic Calogero–Sutherland**

$$\widetilde{H} = -\frac{1}{2}\sum_{m=1}^{M}\partial_{x_m}^2 + \beta(\beta-1)\sum_{m<m'}^{M}\wp(x_m - x_{m'})$$

(for $\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}$ with simple poles at equal arguments)

and if **BAE** $\quad L\,p_m = 2\pi\,I_m + \varphi_m\,, \quad I_m \in \mathbb{Z}_L\,, \quad 1 \leq m \leq M$

Then as long as we identify $\tilde{p}_m = \lambda(p_m)$ [→ next slide]

$$\Psi_{\boldsymbol{p}}(\boldsymbol{n}) = \sum_{w\in S_M}\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}_w)\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{p}-\tilde{\boldsymbol{p}})\cdot\boldsymbol{n}_w}$$

has

$$E = \textstyle\sum_m \varepsilon(p_m) + \widetilde{U} \qquad \widetilde{U} := \widetilde{E} - \frac{1}{2}\sum_m \widetilde{p}_m^{\,2}$$

$$p_{\mathrm{tot}} = \textstyle\sum_m p_m \bmod 2\pi$$

Connection with eCS
Comments

$$\Psi_{\boldsymbol{p}}(\boldsymbol{n}) = \sum_{w \in S_M} \widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}_w)\, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{p}-\tilde{\boldsymbol{p}})\cdot \boldsymbol{n}_w}$$

quasimomenta

eCS wave function ($\beta = 2$)

plane wave

- $\tilde{p}_m = \lambda(p_m)$ **rapidities** $\lambda(p) = -\zeta^\vee(p) + \frac{\zeta^\vee(\pi)}{\pi}\, p$ [Klabbers JL]

  $2\kappa \times -\frac{1}{2}\cot\frac{p}{2} \xleftarrow{\kappa \to \infty} \lambda(p) \xrightarrow{\kappa \to 0} (p-\pi)/2$

- $E = \sum_m \varepsilon(p_m) + \widetilde{U}$ is **additive** iff $\widetilde{E} = \frac{1}{2}\sum_m \tilde{p}_m^2 + \widetilde{U}$ is so [Klabbers JL]
  (which doesn't seem to be the case)

- $\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}$ will have simple poles **or double zeroes** at equal arguments; latter would be better in HS limit [but see later]

Two magnons
From Lamé to Inozemtsev

- Lamé $\widetilde{H}|_{\beta=2} = -\frac{1}{2}\big(\partial_{x_1}^2 + \partial_{x_2}^2\big) + 2\,\wp(x_1 - x_2)$
  Hermite
  $$\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{x}) = A_\gamma(x_1 - x_2)\,\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol{p}}\cdot\boldsymbol{x}}\,, \qquad A_\gamma(x) := \mathrm{e}^{\mathrm{i}\eta_2\,\kappa\,x\,\gamma/\pi}\,\frac{\sigma(x+\gamma)}{\sigma(x)\,\sigma(\gamma)}$$
- Quasiperiodicity fixes γ in terms of $\widetilde{p}_1 - \widetilde{p}_2$
- Inozemtsev
  $$\Psi_{\boldsymbol{p}}(\boldsymbol{n}) = A_\gamma(n_1 - n_2)\,\mathrm{e}^{\mathrm{i}(p_1 n_1 + p_2 n_2)} + A_\gamma(n_2 - n_1)\,\mathrm{e}^{\mathrm{i}(p_1 n_2 + p_2 n_1)}$$
  $$E(\boldsymbol{p}) = \varepsilon(p_1) + \varepsilon(p_2) + \mathrm{cst}(\kappa)\,\widetilde{U}\,, \qquad \widetilde{U} = -4\kappa^2\,\varepsilon|_{\kappa\rightsquigarrow L\kappa}(\varphi)$$
  provided BAE $\begin{cases} L\,p_{1,2} = 2\pi I_{1,2} \pm \varphi \\ \lambda|_{\kappa\rightsquigarrow L\kappa}(\varphi) = \lambda(p_1) - \lambda(p_2) \end{cases}$ $\quad \varphi := 2\mathrm{i}\kappa\gamma$ scattering phase

Two magnons
Rationalisation & completeness

- Eigenvectors with all $p_m \neq 0$ have **highest weight**
- Together the **BAE** $\begin{cases} L\,p_{1,2} = 2\pi\,I_{1,2} \pm \varphi \\ \lambda|_{\kappa \rightsquigarrow L\,\kappa}(\varphi) = \lambda(p_1) - \lambda(p_2) \end{cases}$

  give an equation **elliptic** in φ
- In elliptic coordinates the latter gives a **polynomial** equation
  - **Very efficient** for numerical solutions
  - Can **count number of solutions**: matches # highest weight
- **Energy function** is **elliptic** in φ too
- **Full spectral problem becomes rational** in elliptic coordinates

Two magnons
Heisenberg limit

[Klabbers JL]

- We have $A_\gamma(x)$

$$\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,\mathrm{sgn}(\mathrm{Re}\,x)\,\varphi/2}}{\sin(\varphi/2)} \xleftarrow{\kappa\to\infty} A_\gamma(x)$$

$\varphi := 2\mathrm{i}\kappa\gamma$
remains finite

so $\quad \Psi_{\boldsymbol{p}}(\boldsymbol{n}) = A_\gamma(n_1-n_2)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{n}} + A_\gamma(n_2-n_1)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{n}_\tau}$

$$\frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{n}+\varphi/2)} + \mathrm{e}^{\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{n}_\tau-\varphi/2)}}{\sin(\varphi/2)}$$

becomes **Bethe's wave function**

- **Energy** becomes **additive**
- **BAE**

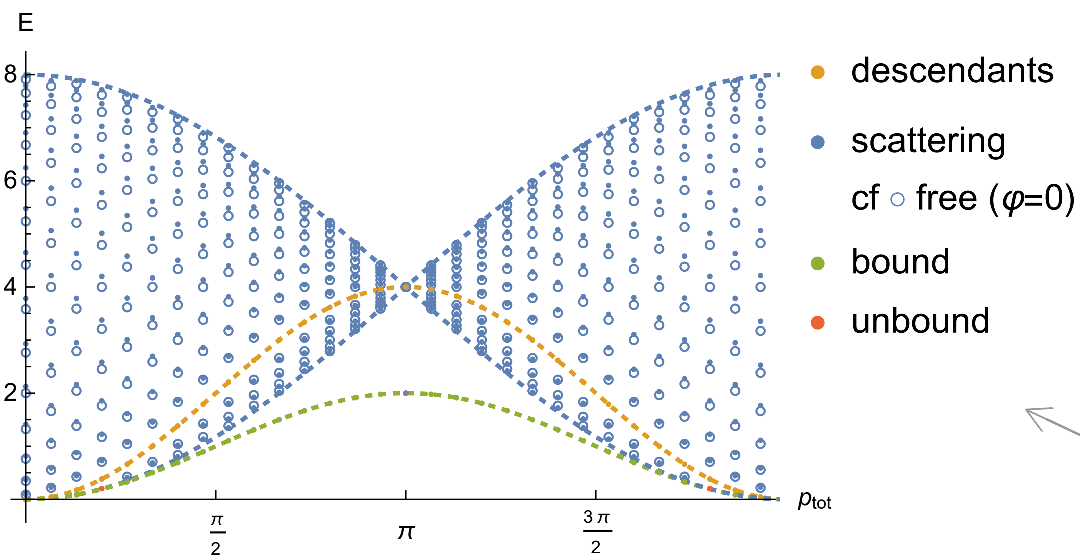
$$\begin{cases} L\,p_{1,2} = 2\pi I_{1,2} \pm \varphi \\ \lambda|_{\kappa\rightsquigarrow L\kappa}(\varphi) = \lambda(p_1) - \lambda(p_2) \end{cases}$$

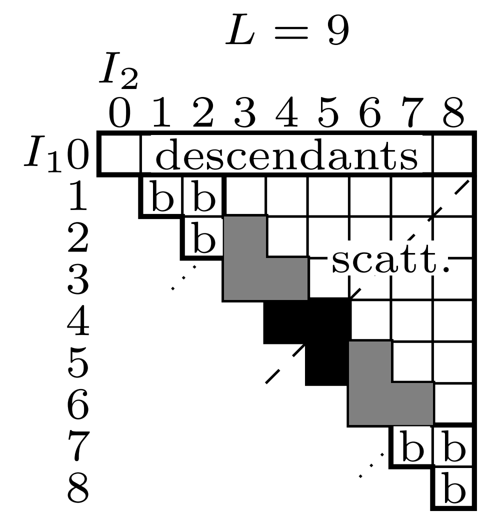
$$2\cot(\varphi/2) = \cot(p_1/2) - \cot(p_2/2)$$

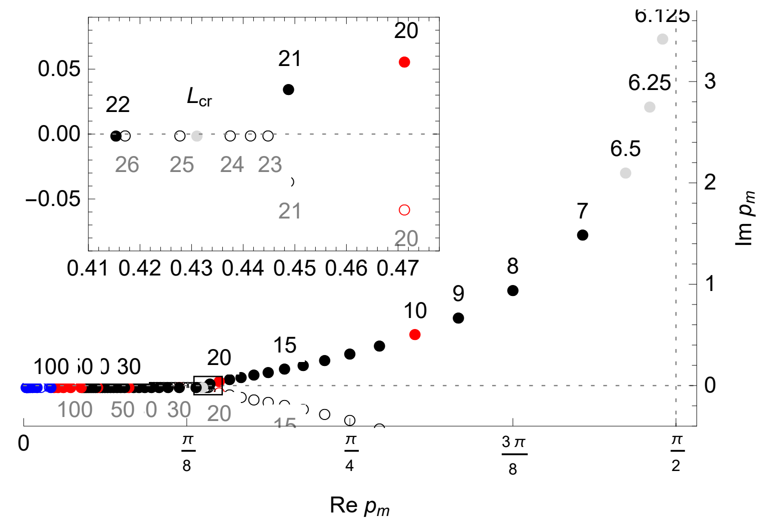
Bethe's equation

Two magnons
Recap of Heisenberg

$$\begin{cases} L\, p_{1,2} = 2\pi I_{1,2} \pm \varphi \\ 2\cot(\varphi/2) = \cot(p_1/2) - \cot(p_2/2) \end{cases}$$







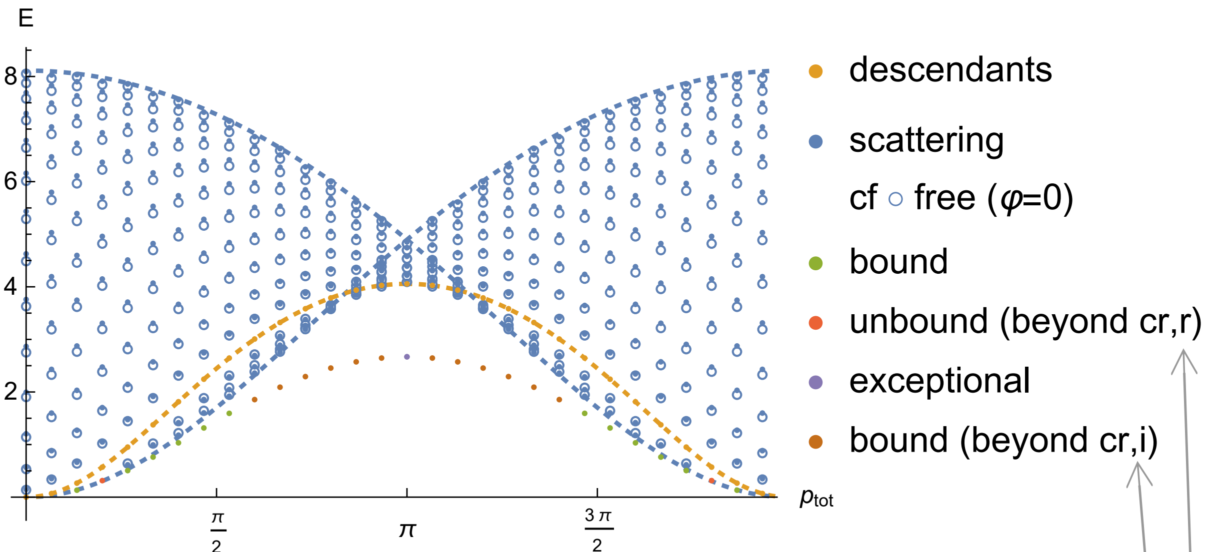
‘critical length’

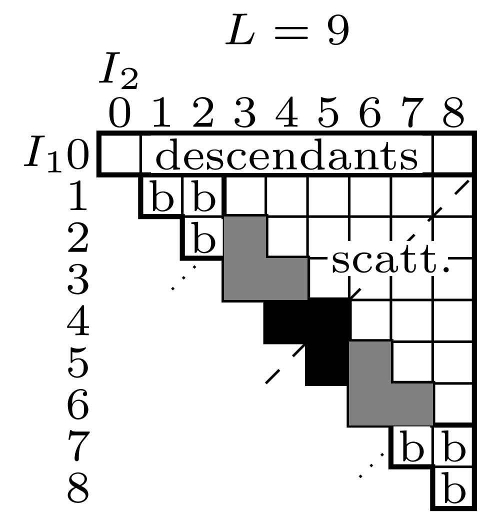
$$L_{\text{cr}}^{(3)} \approx 21.9,\; L_{\text{cr}}^{(5)} \approx 61.3,\; L_{\text{cr}}^{(n)} \approx (\pi n/2)^2$$

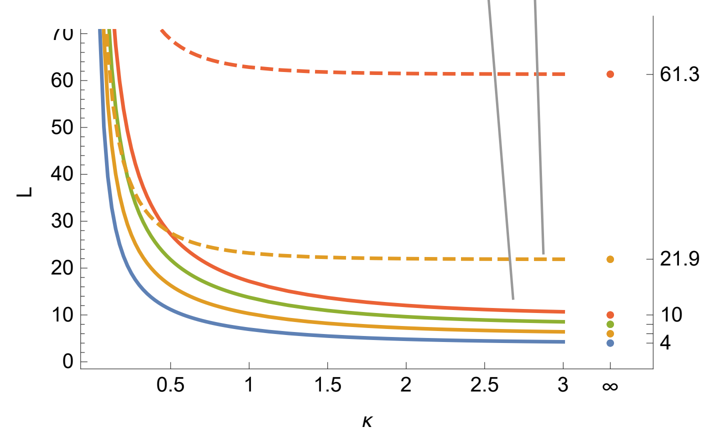
Two magnons
Inozemtsev: critical loci

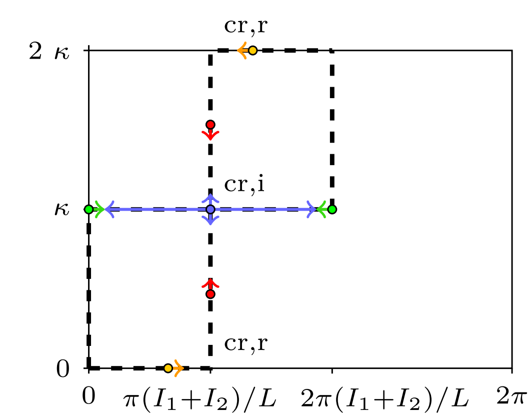
[Klabbers JL]

$$\begin{cases} L\, p_{1,2} = 2\pi\, I_{1,2} \pm \varphi \\ \lambda|_{\kappa \rightsquigarrow L\kappa}(\varphi) = \lambda(p_1) - \lambda(p_2) \end{cases}$$



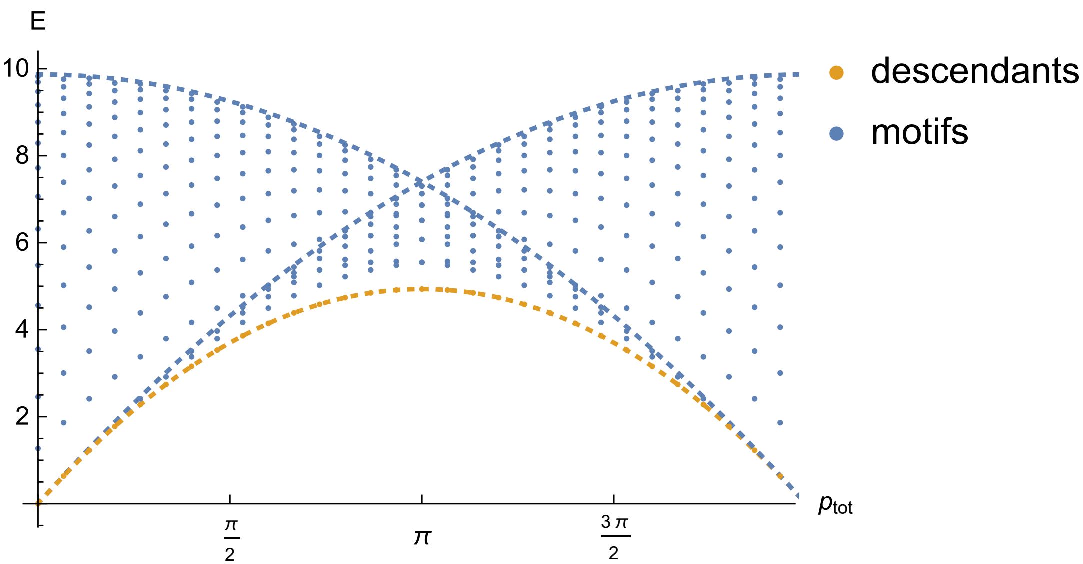


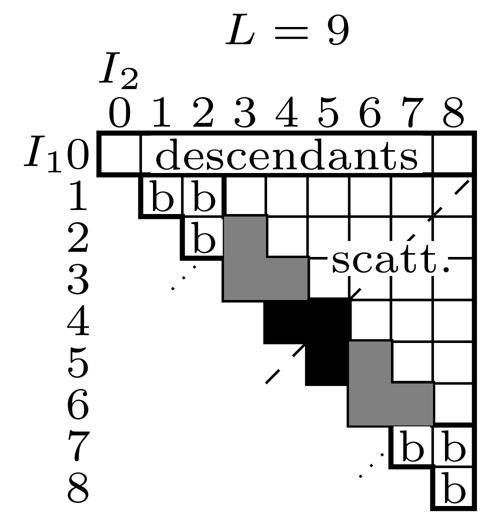


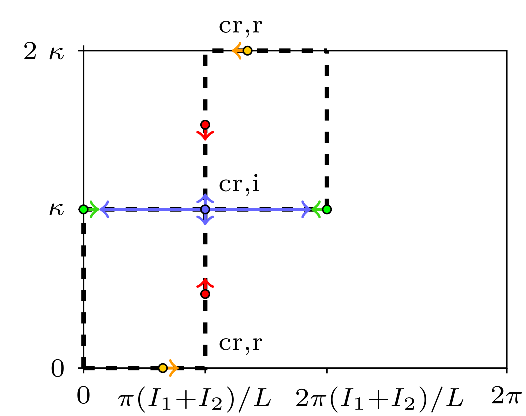


Two magnons
Haldane–Shastry limit (I)

$$\begin{cases} L\,p_{1,2} = 2\pi I_{1,2} \pm \varphi \\ \varphi_{\text{scatt}} \to 0,\ \varphi_{\text{bound}} \to -2\pi I_1 \end{cases}$$







Two magnons
Haldane–Shastry limit (II)

[Klabbers JL]

- We have $$A_\gamma(x) \xrightarrow{\kappa \to 0} \cot\frac{\pi x}{L} - \overbrace{\mathrm{i}\cot\frac{\pi\gamma}{L}}^{=:\,-\xi}$$

  so $$\Psi_{\boldsymbol{p}}(\boldsymbol{n}) = A_\gamma(n_1-n_2)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{n}} + A_\gamma(n_2-n_1)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{n}_\tau}$$

  $$\to \mathrm{ev}\left[\left(\frac{z_{n_1}+z_{n_2}}{z_{n_1}-z_{n_2}}+\xi\right) z_{n_1}^{I_1+j}\, z_{n_2}^{I_2-j} + (n_1 \leftrightarrow n_2)\right]$$

  with **'evaluation'** $\mathrm{ev}:\, z_n \mapsto \mathrm{e}^{2\pi\mathrm{i}n/L}$

- **BAE become trivial**
  $$\begin{cases} L\,p_{1,2} = 2\pi I_{1,2} \pm \varphi \\ \varphi \to 2\pi j,\; j_{\text{scatt}} = 0,\; j_{\text{bound}} = -I_1 \end{cases}$$

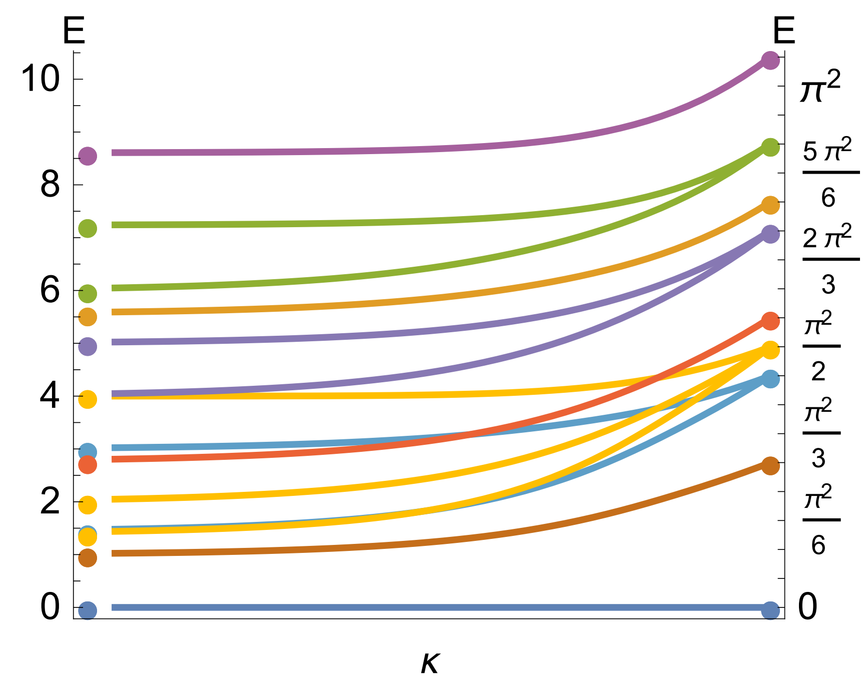
- **Energy** becomes (strictly) **additive**

Scattering states: **Schur expansion of** $\beta = 2$ **Jack** (upon ev)

Conclusion

$$H = \sum_{i<j}^{L} \mathrm{cst}(\kappa)\big(\wp(i-j) + \mathrm{cst}(L,\kappa)\big)\,\frac{\vec{\sigma}_i \cdot \vec{\sigma}_j - 1}{2}$$

- Solution via **relation with $\beta = 2$ elliptic Calogero–Sutherland**
- **'Quasi-additive' energy $E = \sum_m \varepsilon(p_m) + \widetilde{U}$**
- $M = 2$ **limits** in detail, recover known results
- **Spectral problem rationalises**



- **Open questions**
  - $M > 2$?
  - **Quantum integrability** (R, etc) ?
  - **XXZ-like** (q-)analogue ?