Symmetrization map, Casimir elements and Sugawara operators

Alexander Molev

University of Sydney

Kac–Kazhdan conjecture

Kac–Kazhdan conjecture

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

Kac–Kazhdan conjecture

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

Fix a triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$ and consider the Verma modules $M(\lambda)$ with $\lambda \in \widehat{\mathfrak{h}}^*$.

Kac–Kazhdan conjecture

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

Fix a triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$ and consider the Verma modules $M(\lambda)$ with $\lambda \in \widehat{\mathfrak{h}}^*$.

We have an isomorphism of vector spaces

$$M(\lambda) \cong \mathrm{U}(\widehat{\mathfrak{n}}_-)1_\lambda.$$

Hence the character is found by

$$\operatorname{ch} M(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta_{+}^{\mathrm{re}}} (1 - e^{-\alpha})^{-1} \prod_{r=1}^{\infty} (1 - e^{-r\delta})^{-n}.$$

Hence the character is found by

$$\mathrm{ch}\, M(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta_{+}^{\mathrm{re}}} (1 - e^{-\alpha})^{-1} \prod_{r=1}^{\infty} (1 - e^{-r\delta})^{-n}.$$

Suppose that λ is a generic critical weight, $\langle \lambda, K \rangle = -h^{\vee}$.

Hence the character is found by

$$\operatorname{ch} M(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1} \prod_{r=1}^{\infty} (1 - e^{-r\delta})^{-n}.$$

Suppose that λ is a generic critical weight, $\langle \lambda, K \rangle = -h^{\vee}$.

By the Kac–Kazhdan conjecture (1979),

$$\operatorname{ch} L(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1}.$$

Hence the character is found by

$$\operatorname{ch} M(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1} \prod_{r=1}^{\infty} (1 - e^{-r\delta})^{-n}.$$

Suppose that λ is a generic critical weight, $\langle \lambda, K \rangle = -h^{\vee}$.

By the Kac–Kazhdan conjecture (1979),

$$\operatorname{ch} L(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1}.$$

The singular vectors are generated by the Sugawara operators

$$M(\lambda)^{\widehat{\mathfrak{n}}_+} \cong \mathbb{C}[S_{1(r)}, \dots, S_{n(r)} \mid r \geqslant 1];$$

Hence the character is found by

$$\operatorname{ch} M(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1} \prod_{r=1}^{\infty} (1 - e^{-r\delta})^{-n}.$$

Suppose that λ is a generic critical weight, $\langle \lambda, K \rangle = -h^{\vee}$.

By the Kac–Kazhdan conjecture (1979),

$$\operatorname{ch} L(\lambda) = e^{\lambda} \prod_{\alpha \in \Delta_+^{\text{re}}} (1 - e^{-\alpha})^{-1}.$$

The singular vectors are generated by the Sugawara operators

$$M(\lambda)^{\widehat{\mathfrak{n}}_+} \cong \mathbb{C}[S_{1(r)}, \dots, S_{n(r)} \mid r \geqslant 1];$$

[Hayashi 1988, Goodman–Wallach 1989, Feigin–Frenkel 1992].

- The operators $S_{1(r)}, \dots, S_{n(r)}$ with $r \in \mathbb{Z}$ are topological generators of the center of the completed algebra $\widetilde{\mathrm{U}}_{\mathsf{cri}}(\widehat{\mathfrak{g}})$.

- The operators $S_{1(r)}, \dots, S_{n(r)}$ with $r \in \mathbb{Z}$ are topological generators of the center of the completed algebra $\widetilde{\mathrm{U}}_{\mathrm{cri}}(\widehat{\mathfrak{g}})$.
- The operators are obtained from generators $S_1, \dots, S_n$ of a commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ by using a vertex algebra structure.

- The operators $S_{1(r)}, \dots, S_{n(r)}$ with $r \in \mathbb{Z}$ are topological generators of the center of the completed algebra $\widetilde{\mathrm{U}}_{\mathrm{cri}}(\widehat{\mathfrak{g}})$.
- The operators are obtained from generators $S_1, \dots, S_n$ of a commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ by using a vertex algebra structure.
- The subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ (the Feigin–Frenkel center) gives rise to higher order Hamiltonians in the Gaudin model.

- The operators $S_{1(r)}, \dots, S_{n(r)}$ with $r \in \mathbb{Z}$ are topological generators of the center of the completed algebra $\widetilde{\mathrm{U}}_{\text{cri}}(\widehat{\mathfrak{g}})$.
- The operators are obtained from generators $S_1, \dots, S_n$ of a commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ by using a vertex algebra structure.
- The subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ (the Feigin–Frenkel center) gives rise to higher order Hamiltonians in the Gaudin model.
- Applying homomorphisms $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g})$ one gets commutative subalgebras of $\mathrm{U}(\mathfrak{g})$ thus solving Vinberg's quantization problem.

Symmetrization map

Symmetrization map

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$.

Symmetrization map

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$.

The linear map

$$\varpi : \mathrm{S}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{U}(\mathfrak{g}), \qquad x^n \mapsto x^n \qquad \text{for} \quad x \in \mathfrak{g},$$

is a $\mathfrak{g}$-module isomorphism known as the symmetrization map.

Symmetrization map

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$.

The linear map

$$\varpi : \mathrm{S}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{U}(\mathfrak{g}), \qquad x^n \mapsto x^n \qquad \text{for} \quad x \in \mathfrak{g},$$

is a $\mathfrak{g}$-module isomorphism known as the symmetrization map.

Equivalently,

$$\varpi : x_1 \dots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \dots x_{\sigma(n)}, \qquad x_i \in \mathfrak{g}.$$

Hence we have a vector space isomorphism

$$\varpi : \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{Z}(\mathfrak{g}).$$

Hence we have a vector space isomorphism

$$\varpi : \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{Z}(\mathfrak{g}).$$

Let $\mathfrak{g}$ be simple with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

Hence we have a vector space isomorphism

$$\varpi : \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{Z}(\mathfrak{g}).$$

Let $\mathfrak{g}$ be simple with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

Chevalley isomorphism with the Weyl group invariants:

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{S}(\mathfrak{h})^{W}, \qquad \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \cong \mathbb{C}[P_1, \dots, P_n].$$

Hence we have a vector space isomorphism

$$\varpi : \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{Z}(\mathfrak{g}).$$

Let $\mathfrak{g}$ be simple with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

Chevalley isomorphism with the Weyl group invariants:

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{S}(\mathfrak{h})^{W}, \qquad \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \cong \mathbb{C}[P_1, \dots, P_n].$$

Harish-Chandra isomorphism (use the shifted action of W):

$$\chi : \mathrm{Z}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^{W^{\boldsymbol{\cdot}}}, \qquad w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Type A

Type A

The general linear Lie algebra $\mathfrak{gl}_N$ has basis elements

$E_{ij}, \quad i, j = 1, \dots, N,$

Type A

The general linear Lie algebra $\mathfrak{gl}_N$ has basis elements

$E_{ij}, \quad i,j = 1,\dots,N$, with the commutation relations

$$\big[E_{ij}, E_{kl}\big] = \delta_{kj}\, E_{il} - \delta_{il}\, E_{kj}.$$

Type A

The general linear Lie algebra $\mathfrak{gl}_N$ has basis elements

$E_{ij}, \quad i,j = 1,\dots,N$, with the commutation relations

$$\big[E_{ij}, E_{kl}\big] = \delta_{kj}\, E_{il} - \delta_{il}\, E_{kj}.$$

Consider the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & \dots & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}$$

with entries in the symmetric algebra $\mathrm{S}(\mathfrak{gl}_N)$.

Write

$$\det(u + E) = u^N + \Delta_1 u^{N-1} + \cdots + \Delta_N$$

Write

$$\det(u + E) = u^N + \Delta_1 u^{N-1} + \cdots + \Delta_N$$

and

$$\det(1 - qE)^{-1} = 1 + \sum_{k=1}^{\infty} \Phi_k q^k.$$

Write

$$\det(u+E) = u^N + \Delta_1 u^{N-1} + \cdots + \Delta_N$$

and

$$\det(1-qE)^{-1} = 1 + \sum_{k=1}^{\infty} \Phi_k q^k.$$

We have

$$\mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[\Delta_1, \dots, \Delta_N] = \mathbb{C}[\Phi_1, \dots, \Phi_N].$$

Write

$$\det(u+E) = u^N + \Delta_1\, u^{N-1} + \cdots + \Delta_N$$

and

$$\det(1-qE)^{-1} = 1 + \sum_{k=1}^{\infty} \Phi_k\, q^k.$$

We have

$$\mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}\,[\Delta_1, \dots, \Delta_N] = \mathbb{C}\,[\Phi_1, \dots, \Phi_N].$$

This implies

$$\mathrm{Z}(\mathfrak{gl}_N) = \mathbb{C}\left[\varpi(\Delta_1), \dots, \varpi(\Delta_N)\right] = \mathbb{C}\left[\varpi(\Phi_1), \dots, \varpi(\Phi_N)\right].$$

Explicitly,

$$\varpi(\Delta_m) = \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{N} \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \cdot E_{i_{\sigma(1)}\, i_1} \ldots E_{i_{\sigma(m)}\, i_m}$$

Explicitly,

$$\varpi(\Delta_m) = \frac{1}{m!} \sum_{i_1,\ldots,\, i_m=1}^{N} \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \cdot E_{i_{\sigma(1)}\, i_1} \ldots E_{i_{\sigma(m)}\, i_m}$$

and

$$\varpi(\Phi_m) = \frac{1}{m!} \sum_{i_1,\ldots,\, i_m=1}^{N} \sum_{\sigma\in\mathfrak{S}_m} E_{i_{\sigma(1)}\, i_1} \ldots E_{i_{\sigma(m)}\, i_m}.$$

Explicitly,

$$\varpi(\Delta_m) = \frac{1}{m!} \sum_{i_1,\dots,i_m=1}^{N} \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \cdot E_{i_{\sigma(1)}\,i_1} \dots E_{i_{\sigma(m)}\,i_m}$$

and

$$\varpi(\Phi_m) = \frac{1}{m!} \sum_{i_1,\dots,i_m=1}^{N} \sum_{\sigma\in\mathfrak{S}_m} E_{i_{\sigma(1)}\,i_1} \dots E_{i_{\sigma(m)}\,i_m}.$$

Remark. The traces $\operatorname{tr} E^m$ with $m = 1, \dots, N$ are also algebraically independent generators of $\mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$.

Explicitly,

$$\varpi(\Delta_m) = \frac{1}{m!} \sum_{i_1,\dots,\, i_m=1}^{N} \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \cdot E_{i_{\sigma(1)}\, i_1} \dots E_{i_{\sigma(m)}\, i_m}$$

and

$$\varpi(\Phi_m) = \frac{1}{m!} \sum_{i_1,\dots,\, i_m=1}^{N} \sum_{\sigma\in\mathfrak{S}_m} E_{i_{\sigma(1)}\, i_1} \dots E_{i_{\sigma(m)}\, i_m}.$$

Remark. The traces $\operatorname{tr} E^m$ with $m = 1, \dots, N$ are also algebraically independent generators of $\mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$.

Their images $\varpi(\operatorname{tr} E^m)$ are free generators of $\mathrm{Z}(\mathfrak{gl}_N)$.

Given an N-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_N)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$

Given an N-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_N)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant N.$$

Given an N-tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_N)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij}\, \xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$E_{ii}\, \xi = \lambda_i\, \xi \qquad \text{for} \quad 1 \leqslant i \leqslant N.$$

Any element $z \in \mathrm{Z}(\mathfrak{gl}_N)$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$.

Given an N-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_N)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant N.$$

Any element $z \in \mathrm{Z}(\mathfrak{gl}_N)$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. As a function of the highest weight, $\chi(z)$ is a shifted symmetric polynomial in the variables $\lambda_1, \dots, \lambda_N$.

Given an N-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_N)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant N.$$

Any element $z \in \mathrm{Z}(\mathfrak{gl}_N)$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. As a function of the highest weight, $\chi(z)$ is a shifted symmetric polynomial in the variables $\lambda_1, \dots, \lambda_N$.

It is symmetric in the shifted variables $\lambda_1, \lambda_2 - 1, \dots, \lambda_N - N + 1$.

Elementary shifted symmetric polynomials:

$$e_m^*(\lambda_1, \dots, \lambda_N) = \sum_{i_1 < \dots < i_m} \lambda_{i_1}(\lambda_{i_2} - 1) \dots (\lambda_{i_m} - m + 1).$$

Elementary shifted symmetric polynomials:

$$e_m^*(\lambda_1,\dots,\lambda_N)=\sum_{i_1<\dots<i_m}\lambda_{i_1}(\lambda_{i_2}-1)\dots(\lambda_{i_m}-m+1).$$

Complete shifted symmetric polynomials:

$$h_m^*(\lambda_1,\dots,\lambda_N)=\sum_{i_1\leqslant\dots\leqslant i_m}\lambda_{i_1}(\lambda_{i_2}+1)\dots(\lambda_{i_m}+m-1).$$

Elementary shifted symmetric polynomials:

$$e_m^*(\lambda_1, \ldots, \lambda_N) = \sum_{i_1 < \cdots < i_m} \lambda_{i_1}(\lambda_{i_2} - 1) \ldots (\lambda_{i_m} - m + 1).$$

Complete shifted symmetric polynomials:

$$h_m^*(\lambda_1, \ldots, \lambda_N) = \sum_{i_1 \leqslant \cdots \leqslant i_m} \lambda_{i_1}(\lambda_{i_2} + 1) \ldots (\lambda_{i_m} + m - 1).$$

Remark. The shifted Schur polynomials [OO, 1998] are:

$$s_\mu^*(\lambda_1, \ldots, \lambda_N) = \sum_{\mathrm{sh}(T)=\mu} \prod_{\alpha \in \mu} \big(\lambda_{T(\alpha)} + c(\alpha)\big).$$

The Stirling number of the second kind $\left\{ {m \atop k} \right\}$ counts the number of partitions of the set $\{1, \ldots, m\}$ into k nonempty subsets.

The Stirling number of the second kind $\left\{ {m \atop k} \right\}$ counts the number of partitions of the set $\{1, \dots, m\}$ into k nonempty subsets.

Theorem. For the Harish-Chandra images we have

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{N}{m} \binom{N}{k}^{-1} e_k^*(\lambda_1, \dots, \lambda_N)$$

The Stirling number of the second kind $\left\{ {m \atop k} \right\}$ counts the number of partitions of the set $\{1, \ldots, m\}$ into k nonempty subsets.

Theorem. For the Harish-Chandra images we have

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{N}{m} \binom{N}{k}^{-1} e_k^*(\lambda_1, \ldots, \lambda_N)$$

and

$$\chi : \varpi(\Phi_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{-N}{m} \binom{-N}{k}^{-1} h_k^*(\lambda_1, \ldots, \lambda_N).$$

Types B, C and D

Types B, C and D

The orthogonal Lie algebra $\mathfrak{o}_N$ with $N = 2n$ or $N = 2n + 1$

is the subalgebra of $\mathfrak{gl}_N$ spanned by the elements

$$F_{ij} = E_{ij} - E_{j'i'},$$

where $i' = N - i + 1$.

Types B, C and D

The orthogonal Lie algebra $\mathfrak{o}_N$ with $N = 2n$ or $N = 2n + 1$

is the subalgebra of $\mathfrak{gl}_N$ spanned by the elements

$$F_{ij} = E_{ij} - E_{j'i'},$$

where $i' = N - i + 1$.

The symplectic Lie algebra $\mathfrak{sp}_N$ with $N = 2n$ is spanned by

$$F_{ij} = E_{ij} - \varepsilon_i \, \varepsilon_j \, E_{j'i'},$$

where $\varepsilon_i = -\varepsilon_{n+i} = 1$ for $i = 1, \dots, n$.

Consider the matrix

$$F = \begin{bmatrix} F_{11} & \dots & F_{1N} \\ \vdots & \dots & \vdots \\ F_{N1} & \dots & F_{NN} \end{bmatrix}$$

with entries in $\mathrm{S}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{o}_N$ or $\mathfrak{g} = \mathfrak{sp}_N$.

Consider the matrix

$$F = \begin{bmatrix} F_{11} & \dots & F_{1N} \\ \vdots & \dots & \vdots \\ F_{N1} & \dots & F_{NN} \end{bmatrix}$$

with entries in $\mathrm{S}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{o}_N$ or $\mathfrak{g} = \mathfrak{sp}_N$. Write

$$\det(u + F) = u^{2n} + \Delta_2\, u^{2n-2} + \dots + \Delta_{2n} \qquad \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n},$$

Consider the matrix

$$F = \begin{bmatrix} F_{11} & \dots & F_{1N} \\ \vdots & \dots & \vdots \\ F_{N1} & \dots & F_{NN} \end{bmatrix}$$

with entries in $\mathrm{S}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{o}_N$ or $\mathfrak{g} = \mathfrak{sp}_N$. Write

$$\det(u + F) = u^{2n} + \Delta_2\, u^{2n-2} + \dots + \Delta_{2n} \qquad \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n},$$

and

$$\det(1 - qF)^{-1} = 1 + \sum_{k=1}^{\infty} \Phi_{2k}\, q^{2k} \qquad \text{for} \quad \mathfrak{g} = \mathfrak{o}_N.$$

Given any n-tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$, the corresponding irreducible highest weight representation $L(\lambda)$ of the Lie algebra $\mathfrak{g}$ is generated by a nonzero vector $\xi \in L(\lambda)$

Given any n-tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$, the corresponding irreducible highest weight representation $L(\lambda)$ of the Lie algebra $\mathfrak{g}$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$F_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$F_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Given any n-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n)$, the corresponding irreducible highest weight representation $L(\lambda)$ of the Lie algebra $\mathfrak{g}$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$F_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$F_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Any element $z \in \mathrm{Z}(\mathfrak{g})$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$.

Given any n-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n)$, the corresponding irreducible highest weight representation $L(\lambda)$ of the Lie algebra $\mathfrak{g}$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$F_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$F_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Any element $z \in \mathrm{Z}(\mathfrak{g})$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. As a function of the highest weight, $\chi(z)$ is a shifted invariant polynomial in the variables $\lambda_1, \dots, \lambda_n$.

Theorem. (i) For $\mathfrak{g} = \mathfrak{sp}_{2n}$ the Harish-Chandra images are

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{2n+1}{m} \binom{2n+1}{k}^{-1} \times \, e_k^*(\lambda_1, \dots, \lambda_n, 0, -\lambda_n, \dots, -\lambda_1).$$

Theorem. (i) For $\mathfrak{g} = \mathfrak{sp}_{2n}$ the Harish-Chandra images are

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{2n+1}{m} \binom{2n+1}{k}^{-1}$$

$$\times\ e_k^*(\lambda_1, \dots, \lambda_n, 0, -\lambda_n, \dots, -\lambda_1).$$

(ii) For $\mathfrak{g} = \mathfrak{o}_{2n+1}$ the Harish-Chandra images are

$$\chi : \varpi(\Phi_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{-2n}{m} \binom{-2n}{k}^{-1}$$

$$\times\ h_k^*(\lambda_1, \dots, \lambda_n, -\lambda_n, \dots, -\lambda_1).$$

(iii) For $\mathfrak{g} = \mathfrak{o}_{2n}$ the Harish-Chandra images are

$$\chi : \varpi(\Phi_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{-2n+1}{m} \binom{-2n+1}{k}^{-1}$$

$$\times \Big(\frac{1}{2}\, h_k^*(\lambda_1, \dots, \lambda_{n-1}, -\lambda_n, \dots, -\lambda_1)$$

$$+ \frac{1}{2}\, h_k^*(\lambda_1, \dots, \lambda_n, -\lambda_{n-1}, \dots, -\lambda_1) \Big).$$

(iii) For $\mathfrak{g} = \mathfrak{o}_{2n}$ the Harish-Chandra images are

$$\chi : \varpi(\Phi_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{-2n+1}{m} \binom{-2n+1}{k}^{-1}$$

$$\times \Big(\frac{1}{2} h_k^*(\lambda_1, \dots, \lambda_{n-1}, -\lambda_n, \dots, -\lambda_1) + \frac{1}{2} h_k^*(\lambda_1, \dots, \lambda_n, -\lambda_{n-1}, \dots, -\lambda_1) \Big).$$

Remark. If m is odd, then the elements Δ_m, Φ_m and their images are zero.

Proving the theorems

Proving the theorems

Regard the matrix $E = \big[E_{ij}\big]$ as the element

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}\big(\mathfrak{gl}_N\big).$$

Proving the theorems

Regard the matrix $E = \big[E_{ij}\big]$ as the element

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}\big(\mathfrak{gl}_N\big).$$

Consider the algebra

$$\underbrace{\operatorname{End}\mathbb{C}^N \otimes \ldots \otimes \operatorname{End}\mathbb{C}^N}_{m} \otimes \mathrm{U}\big(\mathfrak{gl}_N\big)$$

Proving the theorems

Regard the matrix $E = \big[E_{ij}\big]$ as the element

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}\big(\mathfrak{gl}_N\big).$$

Consider the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^N \otimes \ldots \otimes \operatorname{End} \mathbb{C}^N}_{m} \otimes \mathrm{U}\big(\mathfrak{gl}_N\big)$$

and for $a = 1, \ldots, m$ set

$$E_a = \sum_{i,j=1}^{N} \underbrace{1 \otimes \ldots \otimes 1}_{a-1} \otimes e_{ij} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{m-a} \otimes E_{ij}.$$

The symmetric group $\mathfrak{S}_m$ acts on the tensor product space

$$\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}$$

by permutations of tensor factors.

The symmetric group $\mathfrak{S}_m$ acts on the tensor product space

$$\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}$$

by permutations of tensor factors.

Denote by $H^{(m)}$ and $A^{(m)}$ the respective images of the

symmetrizer and anti-symmetrizer

$$\frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma \qquad \text{and} \qquad \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn} \sigma \cdot \sigma.$$

The symmetric group $\mathfrak{S}_m$ acts on the tensor product space

$$\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}$$

by permutations of tensor factors.

Denote by $H^{(m)}$ and $A^{(m)}$ the respective images of the

symmetrizer and anti-symmetrizer

$$\frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma \qquad \text{and} \qquad \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn} \sigma \cdot \sigma.$$

We regard $H^{(m)}$ and $A^{(m)}$ as elements of the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^N \otimes \ldots \otimes \operatorname{End} \mathbb{C}^N}_{m}.$$

Observe that

$$\varpi(\Delta_m) = \operatorname{tr} A^{(m)} E_1 \ldots E_m$$

Observe that

$$\varpi(\Delta_m) = \operatorname{tr} A^{(m)} E_1 \ldots E_m$$

and

$$\varpi(\Phi_m) = \operatorname{tr} H^{(m)} E_1 \ldots E_m.$$

Observe that

$$\varpi(\Delta_m) = \operatorname{tr} A^{(m)} E_1 \dots E_m$$

and

$$\varpi(\Phi_m) = \operatorname{tr} H^{(m)} E_1 \dots E_m.$$

On the other hand, it is well-known that

$$\chi : \operatorname{tr} A^{(m)} E_1 (E_2 - 1) \dots (E_m - m + 1) \mapsto e^*_m(\lambda_1, \dots, \lambda_N)$$

Observe that

$$\varpi(\Delta_m) = \operatorname{tr} A^{(m)} E_1 \dots E_m$$

and

$$\varpi(\Phi_m) = \operatorname{tr} H^{(m)} E_1 \dots E_m.$$

On the other hand, it is well-known that

$$\chi : \operatorname{tr} A^{(m)} E_1 (E_2 - 1) \dots (E_m - m + 1) \mapsto e^*_m(\lambda_1, \dots, \lambda_N)$$

and

$$\chi : \operatorname{tr} H^{(m)} E_1 (E_2 + 1) \dots (E_m + m - 1) \mapsto h^*_m(\lambda_1, \dots, \lambda_N).$$

Using the identities

$$x^m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} x(x-1)\ldots(x-k+1),$$

Using the identities

$$x^m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} x(x-1)\dots(x-k+1),$$

we derive

$$\operatorname{tr} A^{(m)} E_1 \dots E_m = \operatorname{tr} A^{(m)} \sum_{k=1}^{m} \left\{ {m \atop k} \right\} E_1(E_2-1)\dots(E_k-k+1).$$

Using the identities

$$x^m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} x(x-1)\dots(x-k+1),$$

we derive

$$\operatorname{tr} A^{(m)} E_1 \dots E_m = \operatorname{tr} A^{(m)} \sum_{k=1}^{m} \left\{ {m \atop k} \right\} E_1(E_2-1)\dots(E_k-k+1).$$

The result for $\varpi(\Delta_m)$ now follows by calculating the partial

traces over the spaces $\operatorname{End}\mathbb{C}^N$ labelled by $k+1,\dots,m$, as

$$\operatorname{tr}_m A^{(m)} = \frac{N-m+1}{m} A^{(m-1)}.$$

Affine Kac–Moody algebras

Affine Kac–Moody algebras

- Replace $\mathrm{S}(\mathfrak{g})$ by the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

Affine Kac–Moody algebras

- Replace $\mathrm{S}(\mathfrak{g})$ by the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.
- Replace $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ by the subalgebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

Affine Kac–Moody algebras

- Replace $\mathrm{S}(\mathfrak{g})$ by the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.
- Replace $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ by the subalgebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

We regard $t^{-1}\mathfrak{g}[t^{-1}] \cong \mathfrak{g}[t,t^{-1}]/\mathfrak{g}[t]$ as the adjoint $\mathfrak{g}[t]$-module.

Affine Kac–Moody algebras

- Replace $\mathrm{S}(\mathfrak{g})$ by the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.
- Replace $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ by the subalgebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

We regard $t^{-1}\mathfrak{g}[t^{-1}] \cong \mathfrak{g}[t,t^{-1}]/\mathfrak{g}[t]$ as the adjoint $\mathfrak{g}[t]$-module.

Note that $T = -d/dt$ is a derivation of the symmetric algebra.

Notation: $X[r] = Xt^r$ for $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

Notation: $X[r] = Xt^r$ for $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

An embedding $\mathrm{S}(\mathfrak{g}) \hookrightarrow \mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ is defined by $X \mapsto X[-1]$.

Notation: $X[r] = Xt^r$ for $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

An embedding $\mathrm{S}(\mathfrak{g}) \hookrightarrow \mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ is defined by $X \mapsto X[-1]$.

The image of any element $P \in \mathrm{S}(\mathfrak{g})$ is denoted by $P[-1]$.

Notation: $X[r] = Xt^r$ for $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

An embedding $\mathrm{S}(\mathfrak{g}) \hookrightarrow \mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ is defined by $X \mapsto X[-1]$.

The image of any element $P \in \mathrm{S}(\mathfrak{g})$ is denoted by $P[-1]$.

Theorem [Raïs–Tauvel 1992, Beilinson–Drinfeld 1997].

If $P_1, \dots, P_n$ are algebraically independent generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$,

then the elements $T^r P_1[-1], \dots, T^r P_n[-1]$ with $r \geqslant 0$ are

algebraically independent generators of $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

Define an invariant bilinear form on $\mathfrak{g}$ by

$$\langle X, Y \rangle = \frac{1}{2h^\vee} \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y),$$

where $h^\vee$ is the dual Coxeter number.

Define an invariant bilinear form on $\mathfrak{g}$ by

$$\langle X, Y\rangle = \frac{1}{2h^\vee}\,\mathrm{tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y),$$

where $h^\vee$ is the dual Coxeter number.

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

Define an invariant bilinear form on $\mathfrak{g}$ by

$$\langle X, Y\rangle = \frac{1}{2h^\vee}\,\mathrm{tr}\,(\mathrm{ad}\,X\,\mathrm{ad}\,Y),$$

where $h^\vee$ is the dual Coxeter number.

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

with the commutation relations

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\,K.$$

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^{\vee}$.

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^\vee$.

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\, v = 0\}.$$

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^\vee$.

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\, v = 0\}.$$

We have $V(\mathfrak{g}) \cong \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ as vector spaces,

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^{\vee}$.

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\, v = 0\}.$$

We have $V(\mathfrak{g}) \cong \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ as vector spaces, and

$\mathfrak{z}(\widehat{\mathfrak{g}})$ is a T-invariant commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

Theorem [Feigin–Frenkel 1992].

There exist elements $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

where $n = \operatorname{rank}\mathfrak{g}$, such that

Theorem [Feigin–Frenkel 1992].

There exist elements $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

where $n = \mathrm{rank}\,\mathfrak{g}$, such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}\,[T^k S_l \mid l = 1, \dots, n, \ \ k \geqslant 0].$$

Theorem [Feigin–Frenkel 1992].

There exist elements $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

where $n = \operatorname{rank}\mathfrak{g}$, such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}\,[T^k S_l \mid l = 1, \dots, n, \;\; k \geqslant 0].$$

We call $S_1, \dots, S_n$ a complete set of Segal–Sugawara vectors.

Theorem [Feigin–Frenkel 1992].

There exist elements $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

where $n = \operatorname{rank}\mathfrak{g}$, such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}\,[T^k S_l \mid l = 1, \dots, n, \ \ k \geqslant 0].$$

We call $S_1, \dots, S_n$ a complete set of Segal–Sugawara vectors.

Note that the symmetrization map

$$\varpi : \mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$$

is not a $\mathfrak{g}[t]$-module homomorphism.

Type A

Type A

Theorem. Each family of elements

$$\phi_m = \sum_{k=1}^{m} \binom{N-k}{m-k} \varpi\big(T^{m-k}\Delta_k[-1]\big)\,1$$

Type A

Theorem. Each family of elements

$$\phi_m = \sum_{k=1}^{m} \binom{N-k}{m-k} \varpi\big(T^{m-k}\Delta_k[-1]\big)\,1$$

and

$$\psi_m = \sum_{k=1}^{m} \binom{N+m-1}{m-k} \varpi\big(T^{m-k}\Phi_k[-1]\big)\,1,$$

Type A

Theorem. Each family of elements

$$\phi_m = \sum_{k=1}^{m} \binom{N-k}{m-k} \varpi\big(T^{m-k}\Delta_k[-1]\big)\, 1$$

and

$$\psi_m = \sum_{k=1}^{m} \binom{N+m-1}{m-k} \varpi\big(T^{m-k}\Phi_k[-1]\big)\, 1,$$

where $m = 1, \dots, N$, is a complete set of Segal–Sugawara vectors for $\mathfrak{gl}_N$

Type A

Theorem. Each family of elements

$$\phi_m = \sum_{k=1}^{m} \binom{N-k}{m-k} \varpi\big(T^{m-k}\Delta_k[-1]\big)\,1$$

and

$$\psi_m = \sum_{k=1}^{m} \binom{N+m-1}{m-k} \varpi\big(T^{m-k}\Phi_k[-1]\big)\,1,$$

where $m = 1, \dots, N$, is a complete set of Segal–Sugawara vectors for $\mathfrak{gl}_N$ [Chervov–Talalaev 2006, Chervov–M. 2009].

Working in the algebra

$$\underbrace{\operatorname{End}\mathbb{C}^N \otimes \ldots \otimes \operatorname{End}\mathbb{C}^N}_{m} \otimes \mathrm{U}\big(\widehat{\mathfrak{gl}}_N\big)$$

Working in the algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{m} \otimes \mathrm{U}\big(\widehat{\mathfrak{gl}}_N\big)$$

we can write

$$\phi_m = \mathrm{tr}\, A^{(m)} \big(T + E_1[-1]\big) \ldots \big(T + E_m[-1]\big)\, 1,$$

Working in the algebra

$$\underbrace{\operatorname{End}\mathbb{C}^N \otimes \ldots \otimes \operatorname{End}\mathbb{C}^N}_{m} \otimes \mathrm{U}\big(\widehat{\mathfrak{gl}}_N\big)$$

we can write

$$\phi_m = \operatorname{tr} A^{(m)}\big(T + E_1[-1]\big)\ldots\big(T + E_m[-1]\big)\,1,$$

$$\psi_m = \operatorname{tr} H^{(m)}\big(T + E_1[-1]\big)\ldots\big(T + E_m[-1]\big)\,1,$$

Working in the algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{m} \otimes \mathrm{U}\big(\widehat{\mathfrak{gl}}_N\big)$$

we can write

$$\phi_m = \mathrm{tr}\, A^{(m)} \big(T + E_1[-1]\big) \ldots \big(T + E_m[-1]\big)\, 1,$$

$$\psi_m = \mathrm{tr}\, H^{(m)} \big(T + E_1[-1]\big) \ldots \big(T + E_m[-1]\big)\, 1,$$

where

$$E[-1] = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij}[-1] \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{U}\big(\widehat{\mathfrak{gl}}_N\big).$$

Working in the algebra

$$\underbrace{\operatorname{End}\mathbb{C}^N \otimes \ldots \otimes \operatorname{End}\mathbb{C}^N}_{m} \otimes \mathrm{U}\big(\widehat{\mathfrak{gl}}_N\big)$$

we can write

$$\phi_m = \operatorname{tr} A^{(m)} \big(T + E_1[-1]\big) \ldots \big(T + E_m[-1]\big)\, 1,$$

$$\psi_m = \operatorname{tr} H^{(m)} \big(T + E_1[-1]\big) \ldots \big(T + E_m[-1]\big)\, 1,$$

where

$$E[-1] = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij}[-1] \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}\big(\widehat{\mathfrak{gl}}_N\big).$$

Remark. Another family: $\operatorname{tr}\big(T + E[-1]\big)^m\, 1$.

Eliminate T to get

$$\phi_m = \operatorname{tr} A^{(m)} \big(T + E_1[-1]\big) \dots \big(T + E_m[-1]\big)\, 1$$

$$= \sum_{\lambda \vdash m} c_\lambda \operatorname{tr} A^{(m)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell],$$

Eliminate T to get

$$\phi_m = \operatorname{tr} A^{(m)} \big(T + E_1[-1]\big) \dots \big(T + E_m[-1]\big)\, 1$$

$$= \sum_{\lambda \vdash m} c_\lambda \operatorname{tr} A^{(m)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell],$$

where the parts of partitions λ are $\lambda_1 \geqslant \cdots \geqslant \lambda_\ell > 0$ and

Eliminate T to get

$$\phi_m = \operatorname{tr} A^{(m)} \big(T + E_1[-1]\big) \dots \big(T + E_m[-1]\big)\, 1$$

$$= \sum_{\lambda \vdash m} c_\lambda \operatorname{tr} A^{(m)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell],$$

where the parts of partitions λ are $\lambda_1 \geqslant \dots \geqslant \lambda_\ell > 0$ and

c_λ is the number of permutations of $\{1, \dots, m\}$ of cycle type λ,

$$c_\lambda = \frac{m!}{1^{k_1}\, k_1! \dots m^{k_m}\, k_m!}, \qquad \lambda = (1^{k_1}\, 2^{k_2} \dots m^{k_m}).$$

Theorem. We have the Segal–Sugawara vectors

$$\phi_m^{\circ} = \sum_{\lambda \vdash m} \binom{N}{\ell}^{-1} c_\lambda \operatorname{tr} A^{(\ell)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell]$$

Theorem. We have the Segal–Sugawara vectors

$$\phi_m^\circ = \sum_{\lambda \vdash m} \binom{N}{\ell}^{-1} c_\lambda \operatorname{tr} A^{(\ell)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell]$$

and

$$\psi_m^\circ = \sum_{\lambda \vdash m} \binom{N+\ell-1}{\ell}^{-1} c_\lambda \operatorname{tr} H^{(\ell)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell].$$

Theorem. We have the Segal–Sugawara vectors

$$\phi_m^\circ = \sum_{\lambda \vdash m} \binom{N}{\ell}^{-1} c_\lambda \operatorname{tr} A^{(\ell)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell]$$

and

$$\psi_m^\circ = \sum_{\lambda \vdash m} \binom{N+\ell-1}{\ell}^{-1} c_\lambda \operatorname{tr} H^{(\ell)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell].$$

For proofs and relations between the families, including MacMahon Master Theorem and Newton Identity, see [Sugawara operators for classical Lie algebras, AMS, 2018], Russian edition is available on the MCCME web site.

Types B, C and D

Types B, C and D

Theorem. (i) The family

$$\phi_{2k} = \sum_{l=1}^{k} \binom{2n-2l+1}{2k-2l} \varpi\big(T^{2k-2l}\Delta_{2l}[-1]\big)\, 1$$

with $k = 1, \dots, n$, is a complete set of Segal–Sugawara vectors

for $\mathfrak{g} = \mathfrak{sp}_{2n}$.

Types B, C and D

Theorem. (i) The family

$$\phi_{2k} = \sum_{l=1}^{k} \binom{2n-2l+1}{2k-2l} \varpi\big(T^{2k-2l}\Delta_{2l}[-1]\big)\,1$$

with $k = 1, \dots, n$, is a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{sp}_{2n}$. (ii) The family

$$\psi_{2k} = \sum_{l=1}^{k} \binom{N+2k-2}{2k-2l} \varpi\big(T^{2k-2l}\,\Phi_{2l}[-1]\big)\,1$$

with $k = 1, \dots, n$, is a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{o}_N$ with $N = 2n+1$.

(iii) The family ψ_{2k} with $k = 1, \dots, n-1$ together with

$$\mathrm{Pf}\, F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}\, \sigma \cdot F_{\sigma(1)\, \sigma(2)'}[-1] \dots F_{\sigma(2n-1)\, \sigma(2n)'}[-1]$$

is a complete set of Segal–Sugawara vectors

for $\mathfrak{g} = \mathfrak{o}_N$ with $N = 2n$.

(iii) The family ψ_{2k} with $k = 1, \dots, n-1$ together with

$$\operatorname{Pf} F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1)\,\sigma(2)'}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1]$$

is a complete set of Segal–Sugawara vectors

for $\mathfrak{g} = \mathfrak{o}_N$ with $N = 2n$.

[Yakimova 2019, M. 2013, 2020].

(iii) The family ψ_{2k} with $k = 1, \dots, n-1$ together with

$$\operatorname{Pf} F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1)\,\sigma(2)'}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1]$$

is a complete set of Segal–Sugawara vectors

for $\mathfrak{g} = \mathfrak{o}_N$ with $N = 2n$.

[Yakimova 2019, M. 2013, 2020].

Remark. These results imply the Feigin–Frenkel theorem for the classical types. Formulas for type G_2 are also known by [M.–Ragoucy–Rozhkovskaya 2016, Yakimova 2019].

Theorem. We have the Segal–Sugawara vectors for even m

$$\phi^{\circ}_{m} = \sum_{\lambda \vdash m} \binom{2n+1}{\ell}^{-1} c_{\lambda} \operatorname{tr} A^{(\ell)} F_1[-\lambda_1] \dots F_{\ell}[-\lambda_{\ell}]$$

for $\mathfrak{g} = \mathfrak{sp}_{2n}$, Theorem. We have the Segal–Sugawara vectors for even m

$$\phi^{\circ}_m = \sum_{\lambda\vdash m} \binom{2n+1}{\ell}^{-1} c_\lambda \operatorname{tr} A^{(\ell)} F_1[-\lambda_1]\dots F_\ell[-\lambda_\ell]$$

for $\mathfrak{g} = \mathfrak{sp}_{2n}$,

and

$$\psi^{\circ}_m = \sum_{\lambda\vdash m} \binom{N+\ell-2}{\ell}^{-1} c_\lambda \operatorname{tr} H^{(\ell)} F_1[-\lambda_1]\dots F_\ell[-\lambda_\ell]$$

for $\mathfrak{g} = \mathfrak{o}_N$.

Affine Harish-Chandra isomorphism

Affine Harish-Chandra isomorphism

The Feigin–Frenkel theorem (1992) provides an isomorphism

$$\mathfrak{f} : \mathfrak{z}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} \mathcal{W}({}^L\mathfrak{g}),$$

Affine Harish-Chandra isomorphism

The Feigin–Frenkel theorem (1992) provides an isomorphism

$$\mathfrak{f} : \mathfrak{z}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} \mathcal{W}({}^L\mathfrak{g}),$$

where the classical $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g})$ is associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$.

Affine Harish-Chandra isomorphism

The Feigin–Frenkel theorem (1992) provides an isomorphism

$$\mathfrak{f} : \mathfrak{z}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} \mathcal{W}({}^L\mathfrak{g}),$$

where the classical $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g})$ is associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$. It is obtained by restriction of the affine Harish-Chandra homomorphism

$$\mathfrak{f} : \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{h}} \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big),$$

Affine Harish-Chandra isomorphism

The Feigin–Frenkel theorem (1992) provides an isomorphism

$$\mathfrak{f} : \mathfrak{z}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} \mathcal{W}({}^L\mathfrak{g}),$$

where the classical $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g})$ is associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$. It is obtained by restriction of the affine Harish-Chandra homomorphism

$$\mathfrak{f} : \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{h}} \to \mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big),$$

the projection modulo the left ideal generated by $t^{-1}\mathfrak{n}_-[t^{-1}]$.

Let $\mu_1, \ldots, \mu_n$ be a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Let $\mu_1, \ldots, \mu_n$ be a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Set $\mu_i[r] = \mu_i\, t^r$ and identify

$$\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big) = \mathbb{C}\left[\mu_1[r], \ldots, \mu_n[r] \mid r < 0\right] =: \mathcal{P}_n.$$

Let $\mu_1, \ldots, \mu_n$ be a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Set $\mu_i[r] = \mu_i \, t^r$ and identify

$$\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big) = \mathbb{C}\left[\mu_1[r], \ldots, \mu_n[r] \mid r < 0\right] =: \mathcal{P}_n.$$

The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is defined by

$$\mathcal{W}(\mathfrak{g}) = \{P \in \mathcal{P}_n \mid V_i \, P = 0, \quad i = 1, \ldots, n\},$$

Let $\mu_1, \dots, \mu_n$ be a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Set $\mu_i[r] = \mu_i \, t^r$ and identify

$$\mathrm{U}\big(t^{-1}\mathfrak{h}[t^{-1}]\big) = \mathbb{C}\left[\mu_1[r], \dots, \mu_n[r] \mid r < 0\right] =: \mathcal{P}_n.$$

The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is defined by

$$\mathcal{W}(\mathfrak{g}) = \{P \in \mathcal{P}_n \mid V_i \, P = 0, \quad i = 1, \dots, n\},$$

the V_i are the screening operators.

Introduce the noncommutative elementary symmetric functions

$$e_m(x_1, \ldots, x_p) = \sum_{i_1 > \cdots > i_m} x_{i_1} \ldots x_{i_m}$$

Introduce the noncommutative elementary symmetric functions

$$e_m(x_1, \ldots, x_p) = \sum_{i_1 > \cdots > i_m} x_{i_1} \ldots x_{i_m}$$

and the noncommutative complete symmetric functions

$$h_m(x_1, \ldots, x_p) = \sum_{i_1 \leqslant \cdots \leqslant i_m} x_{i_1} \ldots x_{i_m}.$$

Type A

Type A

Theorem.

The Harish-Chandra images of

the Segal–Sugawara vectors $\phi_m, \psi_m \in \mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ are given by

Type A

Theorem.

The Harish-Chandra images of

the Segal–Sugawara vectors $\phi_m, \psi_m \in \mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ are given by

$$\mathfrak{f} : \phi_m \mapsto e_m\big(T + \mu_1[-1], \ldots, T + \mu_N[-1]\big)\, 1$$

Type A

Theorem.

The Harish-Chandra images of

the Segal–Sugawara vectors $\phi_m, \psi_m \in \mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ are given by

$$\mathfrak{f} : \phi_m \mapsto e_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big)\, 1$$

and

$$\mathfrak{f} : \psi_m \mapsto h_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big)\, 1.$$

Type A

Theorem.

The Harish-Chandra images of

the Segal–Sugawara vectors $\phi_m, \psi_m \in \mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ are given by

$$\mathfrak{f} : \phi_m \mapsto e_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big)\, 1$$

and

$$\mathfrak{f} : \psi_m \mapsto h_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big)\, 1.$$

[Chervov–M. 2009].

Types B, C and D

Types B, C and D

Theorem. (i) If $\mathfrak{g} = \mathfrak{sp}_{2n}$ then the image of ϕ_{2k} under $\mathfrak{f}$ is

$$e_{2k}\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T, T - \mu_n[-1], \dots, T - \mu_1[-1]\big)\,1.$$

Types B, C and D

Theorem. (i) If $\mathfrak{g} = \mathfrak{sp}_{2n}$ then the image of ϕ_{2k} under $\mathfrak{f}$ is

$$e_{2k}\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T, T - \mu_n[-1], \dots, T - \mu_1[-1]\big)\, 1.$$

(ii) If $\mathfrak{g} = \mathfrak{o}_{2n+1}$ then the image of ψ_{2k} under $\mathfrak{f}$ is

$$h_{2k}\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T - \mu_n[-1], \dots, T - \mu_1[-1]\big)\, 1.$$

(iii) If $\mathfrak{g} = \mathfrak{o}_{2n}$ then the image of ψ_{2k} under $\mathfrak{f}$ is

$$\frac{1}{2}\, h_{2k}\big(T+\mu_1[-1],\dots,T+\mu_{n-1}[-1],T-\mu_n[-1],\dots,T-\mu_1[-1]\big)\,1$$

$$+\,\frac{1}{2}\, h_{2k}\big(T+\mu_1[-1],\dots,T+\mu_n[-1],T-\mu_{n-1}[-1],\dots,T-\mu_1[-1]\big)\,1.$$

(iii) If $\mathfrak{g} = \mathfrak{o}_{2n}$ then the image of ψ_{2k} under $\mathfrak{f}$ is

$$\frac{1}{2}\, h_{2k}\big(T + \mu_1[-1], \dots, T + \mu_{n-1}[-1], T - \mu_n[-1], \dots, T - \mu_1[-1]\big)\, 1$$

$$+\, \frac{1}{2}\, h_{2k}\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T - \mu_{n-1}[-1], \dots, T - \mu_1[-1]\big)\, 1.$$

Moreover,

$$\mathfrak{f} : \operatorname{Pf} F[-1] \mapsto \big(\mu_1[-1] - T\big) \dots \big(\mu_n[-1] - T\big)\, 1.$$

(iii) If $\mathfrak{g} = \mathfrak{o}_{2n}$ then the image of ψ_{2k} under $\mathfrak{f}$ is

$$\frac{1}{2}\, h_{2k}\big(T + \mu_1[-1], \dots, T + \mu_{n-1}[-1], T - \mu_n[-1], \dots, T - \mu_1[-1]\big)\, 1$$

$$+\, \frac{1}{2}\, h_{2k}\big(T + \mu_1[-1], \dots, T + \mu_n[-1], T - \mu_{n-1}[-1], \dots, T - \mu_1[-1]\big)\, 1.$$

Moreover,

$$\mathfrak{f} : \operatorname{Pf} F[-1] \mapsto \big(\mu_1[-1] - T\big) \dots \big(\mu_n[-1] - T\big)\, 1.$$

[M.–Mukhin 2014, Rozhkovskaya 2014].

Screening operators

Screening operators

For $\mathcal{W}(\mathfrak{gl}_N)$ the operators $V_1, \ldots, V_{N-1}$ are

Screening operators

For $\mathcal{W}(\mathfrak{gl}_N)$ the operators $V_1, \ldots, V_{N-1}$ are

$$V_i = \sum_{r=0}^{\infty} V_{i\,r} \left(\frac{\partial}{\partial \mu_i[-r-1]} - \frac{\partial}{\partial \mu_{i+1}[-r-1]} \right)$$

Screening operators

For $\mathcal{W}(\mathfrak{gl}_N)$ the operators $V_1, \dots, V_{N-1}$ are

$$V_i = \sum_{r=0}^{\infty} V_{i\,r} \left(\frac{\partial}{\partial \mu_i[-r-1]} - \frac{\partial}{\partial \mu_{i+1}[-r-1]} \right)$$

with

$$\sum_{r=0}^{\infty} V_{i\,r}\, z^r = \exp \sum_{m=1}^{\infty} \frac{\mu_i[-m] - \mu_{i+1}[-m]}{m}\, z^m.$$

For $\mathcal{W}(\mathfrak{o}_N)$ and $\mathcal{W}(\mathfrak{sp}_{2n})$ the operators $V_1, \dots, V_{n-1}$

are given by the above formulas, while

For $\mathcal{W}(\mathfrak{o}_N)$ and $\mathcal{W}(\mathfrak{sp}_{2n})$ the operators $V_1, \dots, V_{n-1}$

are given by the above formulas, while

$$V_n = \sum_{r=0}^{\infty} V_{n\,r} \, \frac{\partial}{\partial \mu_n[-r-1]}$$

For $\mathcal{W}(\mathfrak{o}_N)$ and $\mathcal{W}(\mathfrak{sp}_{2n})$ the operators $V_1, \dots, V_{n-1}$ are given by the above formulas, while

$$V_n = \sum_{r=0}^{\infty} V_{n\,r} \, \frac{\partial}{\partial \mu_n[-r-1]}$$

with

$$\sum_{r=0}^{\infty} V_{n\,r} \, z^r = \exp \sum_{m=1}^{\infty} \frac{\mu_n[-m]}{m} \, z^m$$

for type B_n, and by similar formulas in types C_n and D_n.