Symmetric functions from vertex models

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Part 1 - https://arxiv.org/abs/2007.10886
Part 2 - https://arxiv.org/abs/2003.14260 (j.w. Matteo Mucciconi)

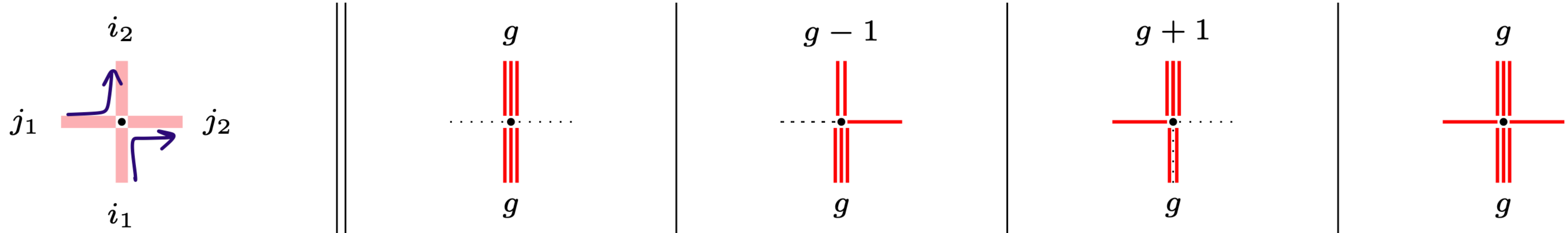
Part 1

Determinantal summation identity for spin Hall-Littlewood functions, and a formula for ASEP

First, we focus on the **spin Hall-Littlewood functions** - a simpler example.
[Borodin 2014], [Borodin-P. 2016]

As an application we get a new determinantal summation formula. [P. 2020]

Spin Hall-Littlewood vertex weights



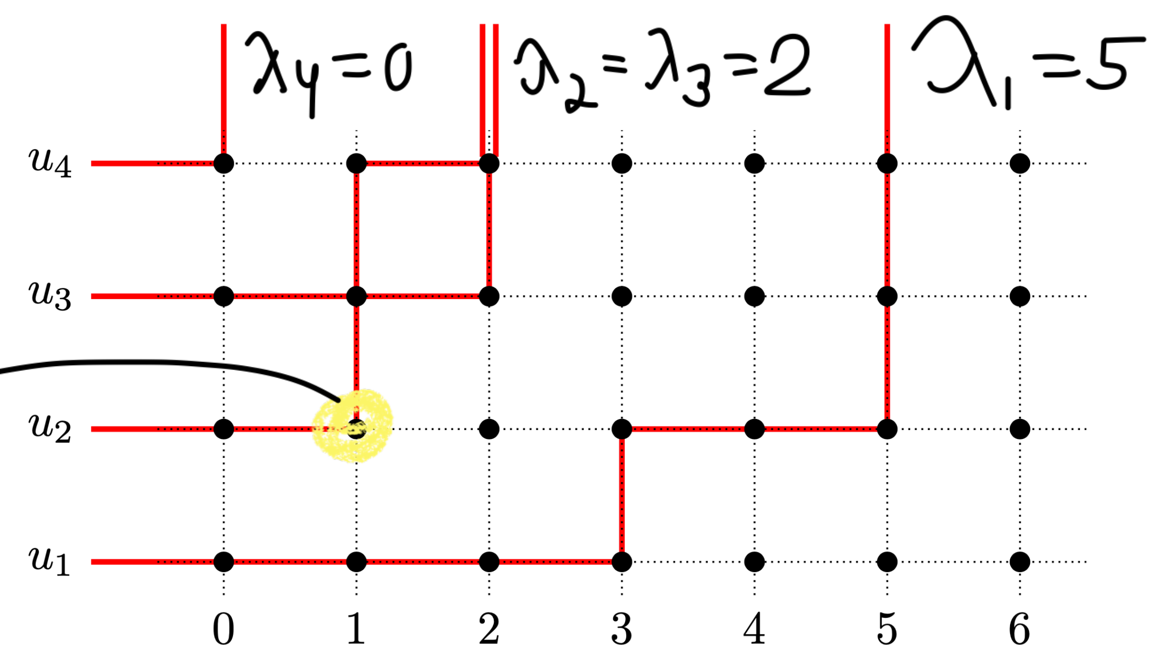
$w_{u,s}(i_1, j_1; i_2, j_2)$	$\dfrac{1 - suq^g}{1 - su}$	$\dfrac{u(1 - s^2q^{g-1})}{1 - su}$	$\dfrac{1 - q^{g+1}}{1 - su}$	$\dfrac{u - sq^g}{1 - su}$

Symmetric functions

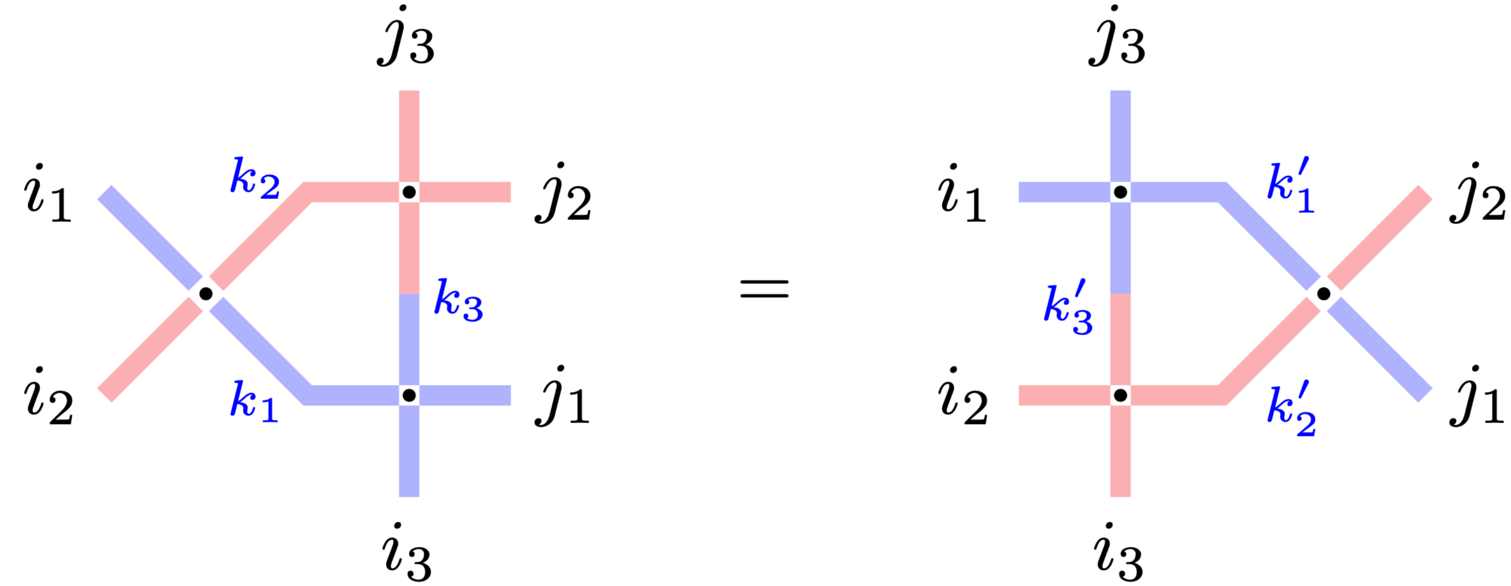
$$F_\lambda(u_1, \dots, u_N)$$

$$\lambda = (\lambda_1 \geq \dots \geq \lambda_N \geq 0)$$

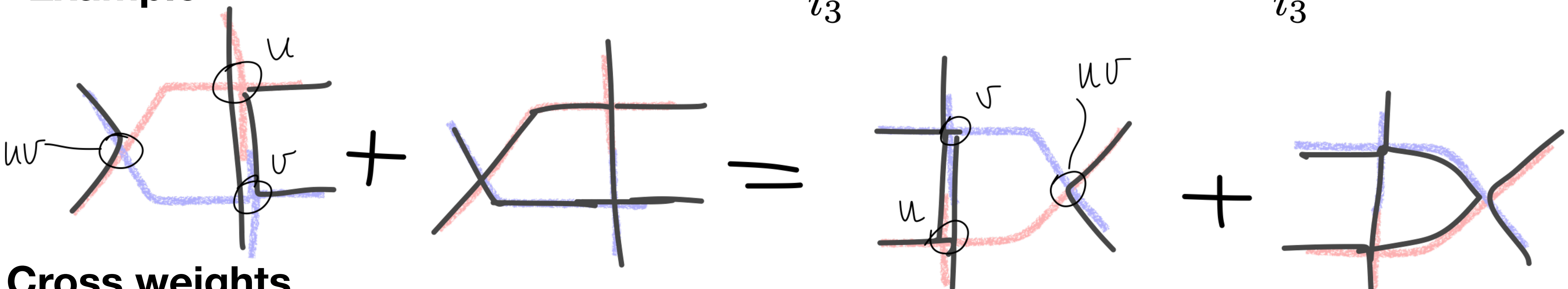
weight $w_{u_2\xi_1, s_1}$



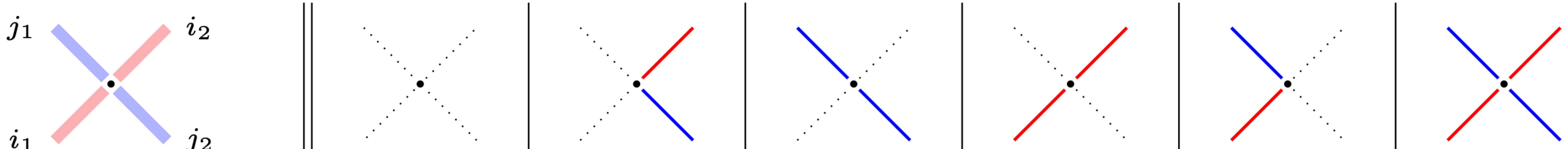
Yang-Baxter equation



Example

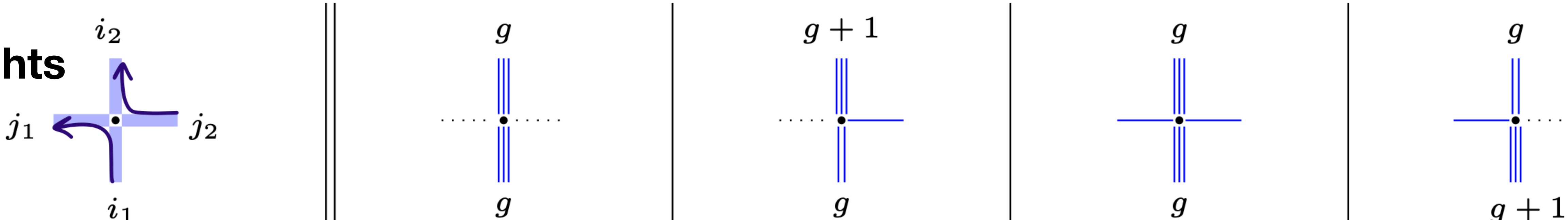


Cross weights



$R_z(i_1, j_1; i_2, j_2)$	1	$\frac{(1-q)z}{1-z}$	$\frac{1-qz}{1-z}$	$\frac{1-qz}{1-z}$	$\frac{1-q}{1-z}$	q

Dual weights



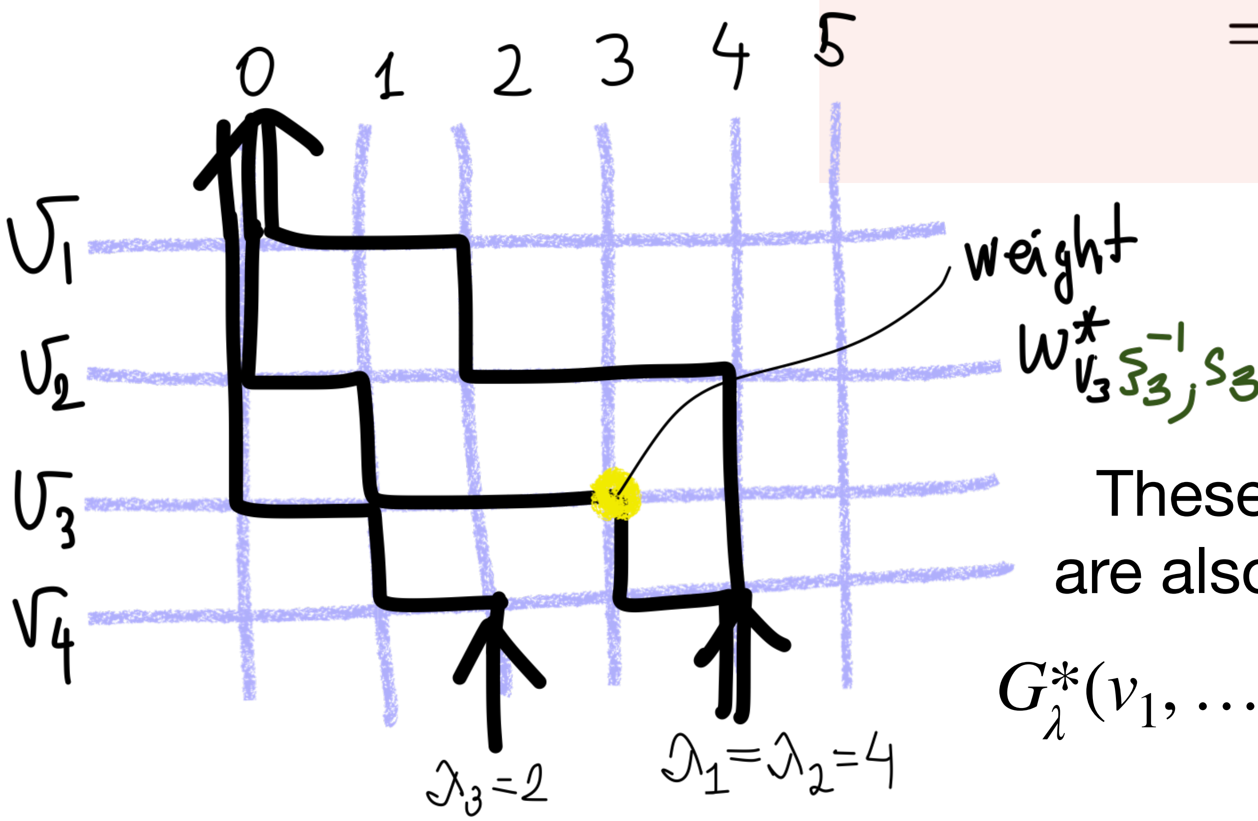
$w^*_{v,s}(i_1, j_1; i_2, j_2)$	$\dfrac{1 - svq^g}{1 - sv}$	$\dfrac{v(1 - q^{g+1})}{1 - sv}$	$\dfrac{v - sq^g}{1 - sv}$	$\dfrac{1 - s^2q^g}{1 - sv}$

Cauchy identity

[Borodin-P. 2016]

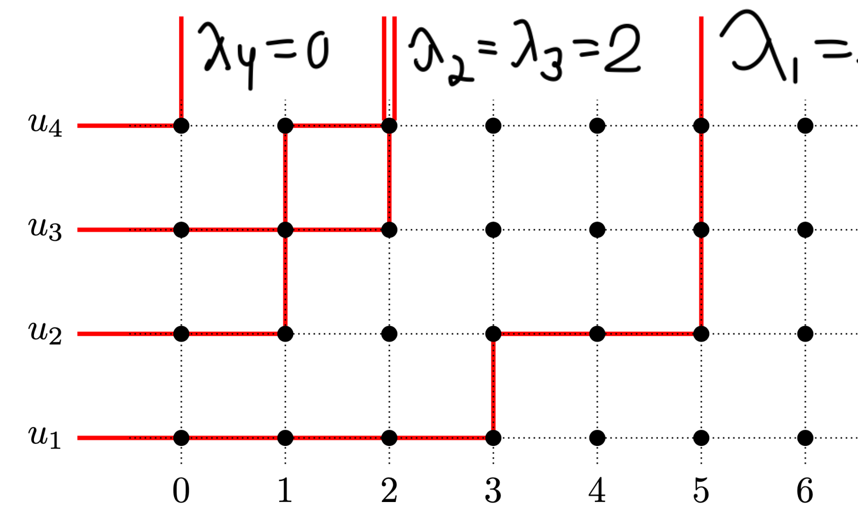
$$\sum_{\lambda} \mathsf{F}_\lambda(u_1,\ldots,u_N)\,\mathsf{G}^*_\lambda(v_1,\ldots,v_M) = \frac{(q;q)_N}{\prod_{i=1}^N (1-s_0\xi_0 u_i)} \prod_{i=1}^N \prod_{j=1}^M \frac{1-qu_iv_j}{1-u_iv_j}$$

$|u_iv_j|<c$



These functions are also symmetric

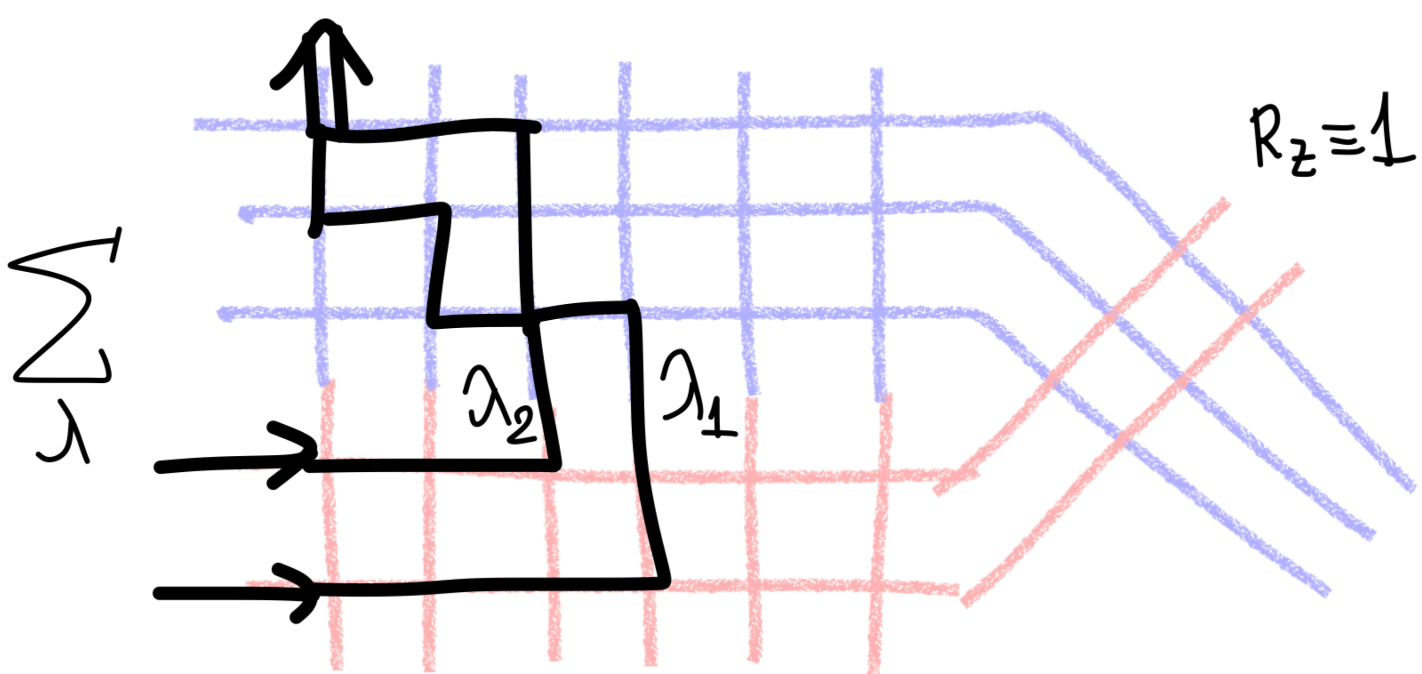
$G^*_\lambda(v_1,\ldots,v_M)$



$F_\lambda(u_1,\ldots,u_N)$

(only one nontrivial configuration)

Proof of the identity



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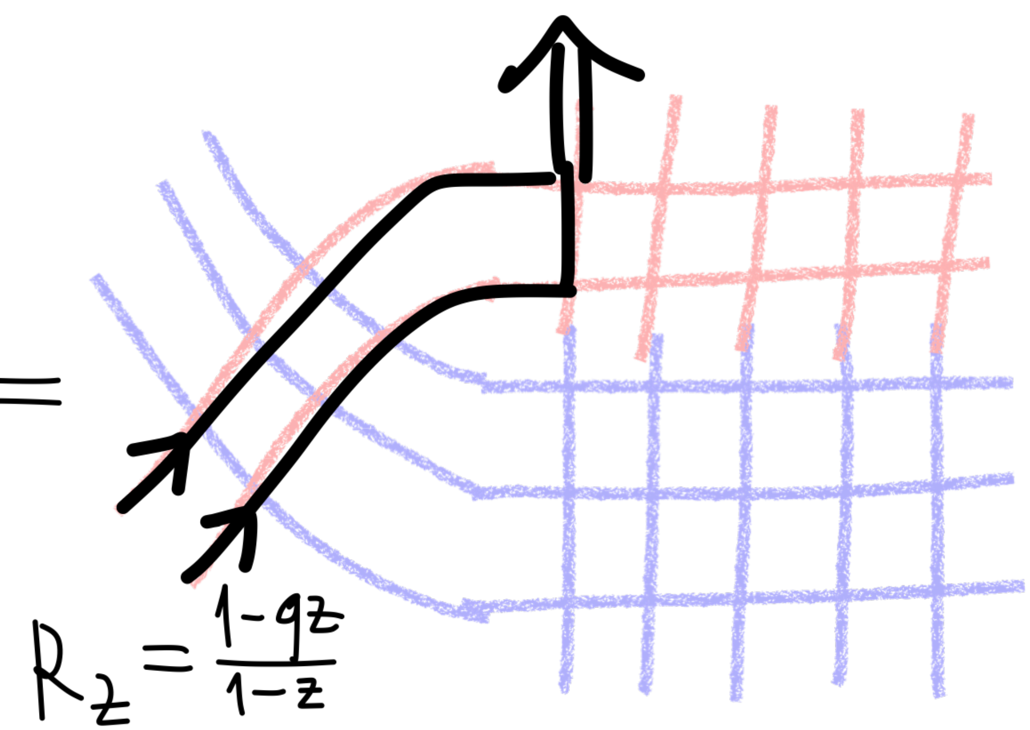


Symmetrization formulas

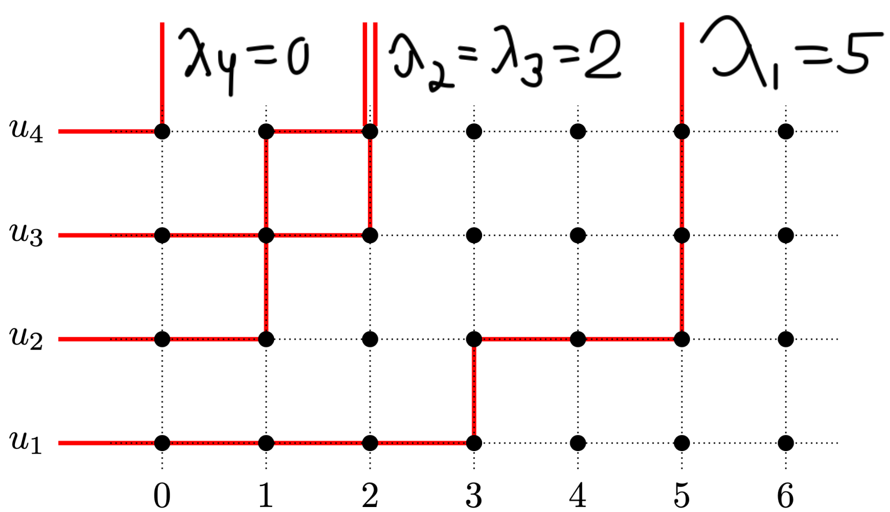
[Borodin-P. 2016]

$$\varphi_k(u) := \frac{1-q}{1-s_k\xi_k u}\prod_{j=0}^{k-1}\frac{\xi_j u - s}{1-s_j\xi_j u}$$

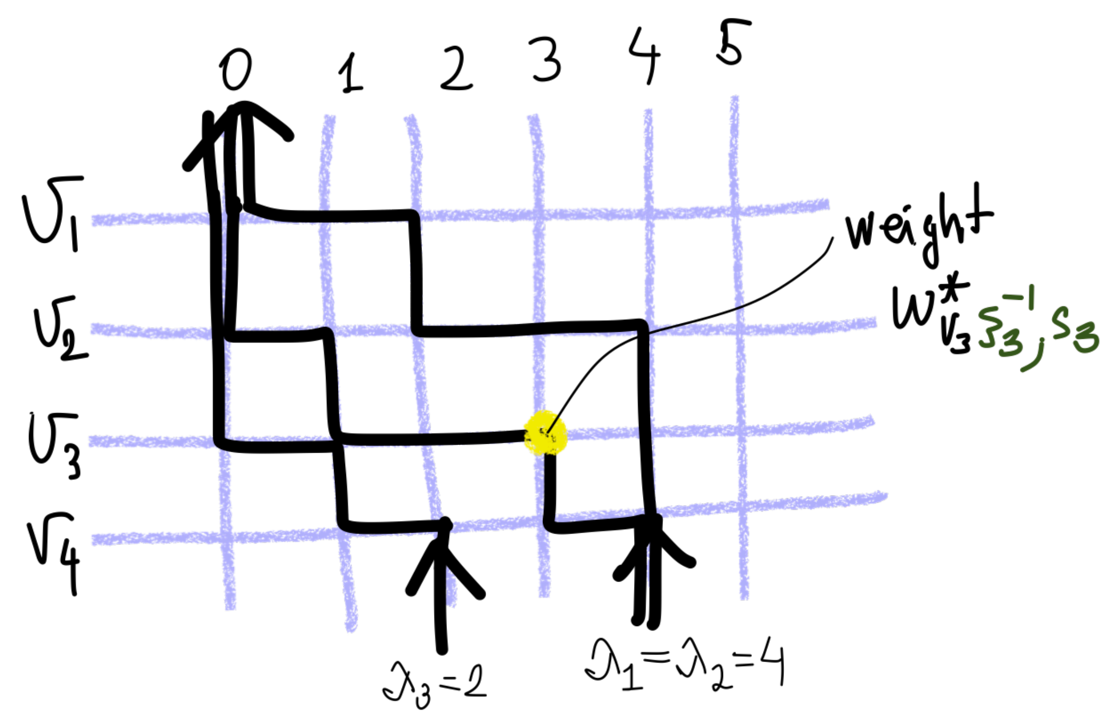
$$\mathsf{F}_\lambda(u_1,\ldots,u_N) = \sum_{\sigma\in S_N}\sigma\left(\prod_{1\le i<j\le N}\frac{u_i-qu_j}{u_i-u_j}\prod_{i=1}^{N}\varphi_{\lambda_i}(u_i)\right)$$

$$\mathsf{G}^*_\lambda(v_1,\ldots,v_K) = \frac{(q;q)_N}{(q;q)_{m_0(\lambda)}(q;q)_{K-\ell(\lambda)}}\prod_{r\ge 1}\frac{(s_r^2;q)_{m_r(\lambda)}}{(q;q)_{m_r(\lambda)}}\sum_{\sigma\in S_K}\sigma\Bigg(\prod_{1\le i<j\le K}\frac{v_i-qv_j}{v_i-v_j}$$

$$\times\prod_{i=1}^{\ell(\lambda)}\frac{v_i}{v_i-s_0\xi_0}\prod_{i=\ell(\lambda)+1}^{K}(1-v_iq^{m_0(\lambda)}s_0/\xi_0)\prod_{j=1}^{K}\left(\varphi_{\lambda_j}(v_j)\Big|_{\xi_x\to\xi_x^{-1}}\right)\Bigg)$$



$F_\lambda(u_1,\ldots,u_N)$



$G^*_\lambda(v_1,\ldots,v_M)$

Particular cases of spin Hall-Littlewood functions F_λ

- $s_k \equiv 0$ - Hall-Littlewood symmetric polynomials (replace q by t)

- $s_k \equiv 0,\ q = 0$ - Schur symmetric polynomials. Cauchy identity is well-known:
$$\sum_\lambda s_\lambda(x_1, \ldots, x_N) s_\lambda(y_1, \ldots, y_M) = \prod_{i,j} \frac{1}{1 - x_i y_j}\,.$$

- (*remark*) Reduce as $s_k = 0,\ \xi_k = 1\ (k \geq 1)$, and $s_0\xi_0 \to 0,\ s_0/\xi_0 \to q^{1-N}$ to the Hall-Littlewood limit of **interpolation Macdonald polynomials**, namely, $I_\lambda(u_1, \ldots, u_N; 0, 1/q)$ **[P. 2020]**. This also *extends to coloured* (higher rank) setting to *nonsymmetric* interpolation HL polynomials.

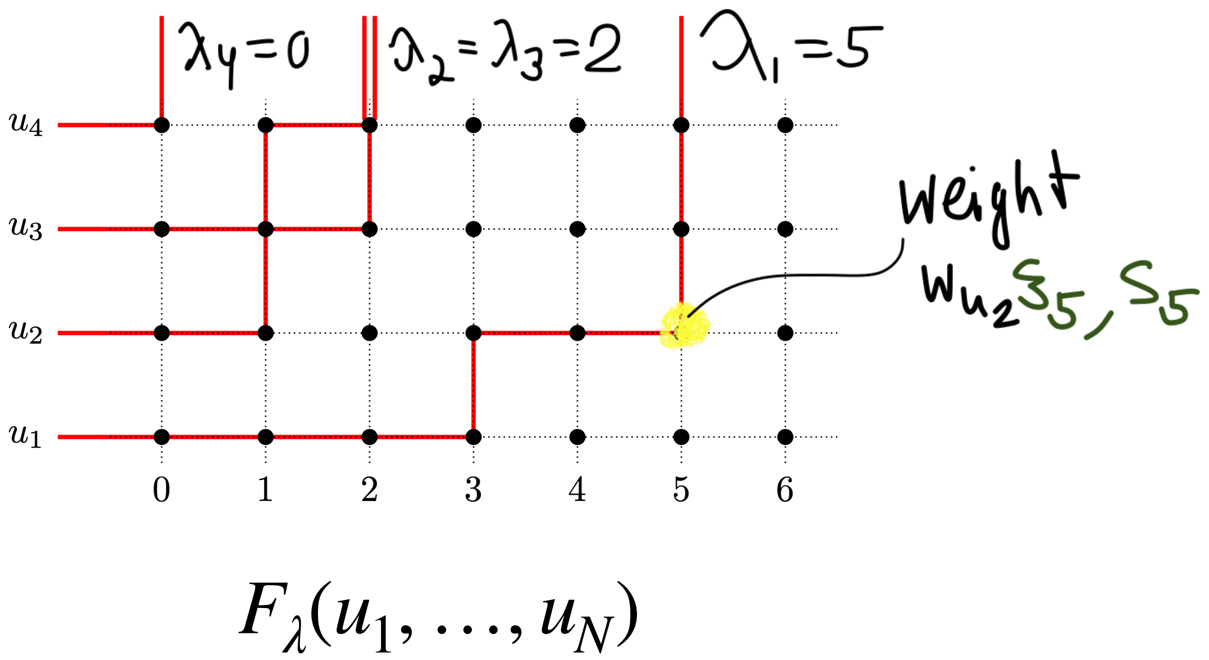
- (*remark*) $q = 0$, Grothendieck like polynomials

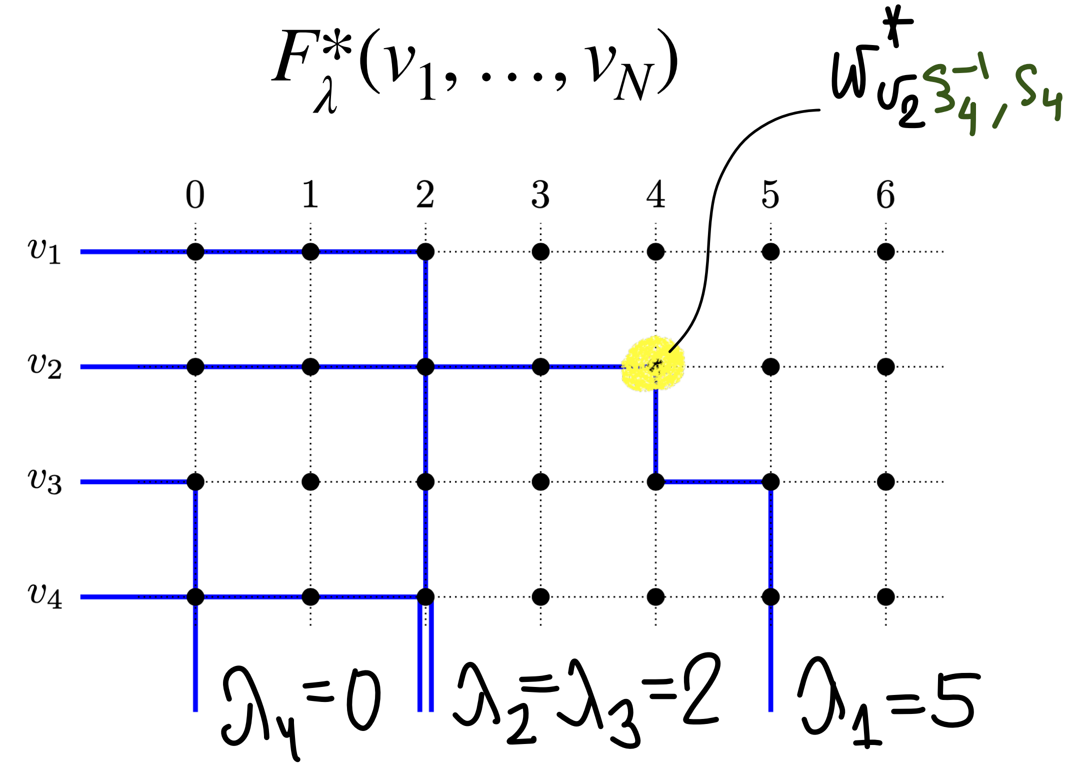
- (***main point***) $s_k \equiv 1/\sqrt{q},\ \xi_k \equiv 1$ - eigenfunctions of the ASEP (**Asymmetric simple exclusion process**).
In ASEP specialization, at most one vertical path is allowed per edge. So Cauchy identity **does not** degenerate to ASEP because G^*_λ does not make sense.

Using Yang-Baxter equation, we obtain a Cauchy-like identity which makes sense in the ASEP specialization.

Refined Cauchy identity

Hall-Littlewood case: **[Wheeler-Zinn Justin 2015]**





$(a;q)_k=(1-a)(1-aq)\ldots(1-aq^{k-1})$

$m_0(\lambda)=N-\ell(\lambda)=\#\{\text{zeroes in }\lambda\}$

$|u_i v_j|<c\ \forall i,j$

[P. 2020] For any $\gamma\neq 0$ we have

$$\sum_{\lambda=(\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_N\ge 0)}\frac{(\gamma q;q)_{m_0(\lambda)}(\gamma^{-1}s_0^2;q)_{m_0(\lambda)}}{(q;q)_{m_0(\lambda)}(s_0^2;q)_{m_0(\lambda)}}F_\lambda(u_1,\ldots,u_N)\,F^*_\lambda(v_1,\ldots,v_N)$$

$$=\prod_{j=1}^N\frac{1}{(1-s_0\xi_0u_j)(1-\xi_0^{-1}s_0v_j)}\frac{\prod_{i,j=1}^N(1-qu_iv_j)}{\prod_{1\le i<j\le N}(u_i-u_j)(v_i-v_j)}$$

$$\times\det\left[\frac{(1-\gamma)(q-\gamma^{-1}s_0^2)(1-u_iv_j)+(1-q)(1-\xi_0s_0u_i)(1-\xi_0^{-1}s_0v_j)}{(1-u_iv_j)(1-qu_iv_j)}\right]_{i,j=1}^N.$$

$$\sum_{\lambda=(\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_N\ge 0)} \frac{(\gamma q;q)_{m_0(\lambda)}(\gamma^{-1}s_0^2;q)_{m_0(\lambda)}}{(q;q)_{m_0(\lambda)}(s_0^2;q)_{m_0(\lambda)}} F_\lambda(u_1,\ldots,u_N)\, F^*_\lambda(v_1,\ldots,v_N)$$
$$= \prod_{j=1}^N \frac{1}{(1-s_0\xi_0 u_j)(1-\xi_0^{-1}s_0 v_j)} \frac{\prod_{i,j=1}^N(1-qu_iv_j)}{\prod_{1\le i<j\le N}(u_i-u_j)(v_i-v_j)}$$
$$\times \det\left[\frac{(1-\gamma)(q-\gamma^{-1}s_0^2)(1-u_iv_j)+(1-q)(1-\xi_0 s_0 u_i)(1-\xi_0^{-1}s_0v_j)}{(1-u_iv_j)(1-qu_iv_j)}\right]_{i,j=1}^N.$$

In particular, for $\gamma = 1$:

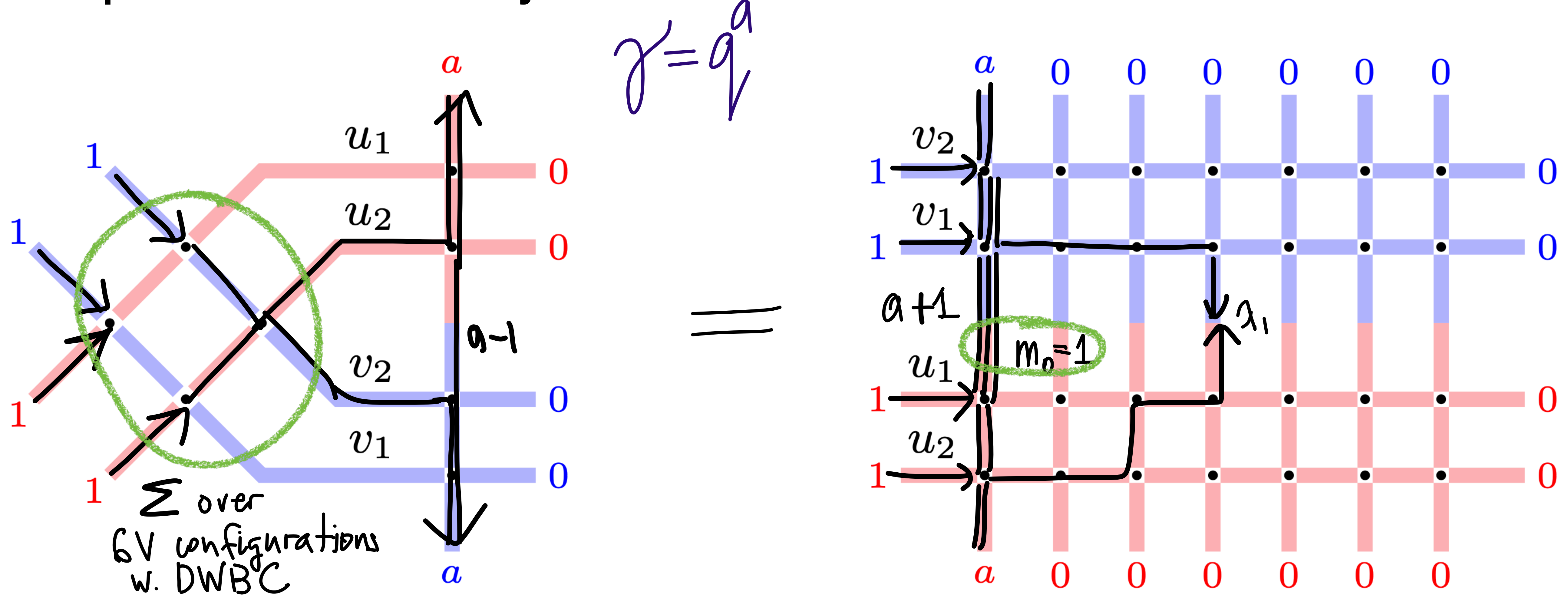
$$\sum_\lambda F_\lambda(u_1,\ldots,u_N)\, F^*_\lambda(v_1,\ldots,v_N) = \frac{(1-q)^N\prod_{i,j=1}^N(1-qu_iv_j)}{\prod_{1\le i<j\le N}(u_i-u_j)(v_i-v_j)} \underbrace{\det\left[\frac{1}{(1-u_iv_j)(1-qu_iv_j)}\right]_{i,j=1}^N}_{\text{Izergin-Korepin det}}$$

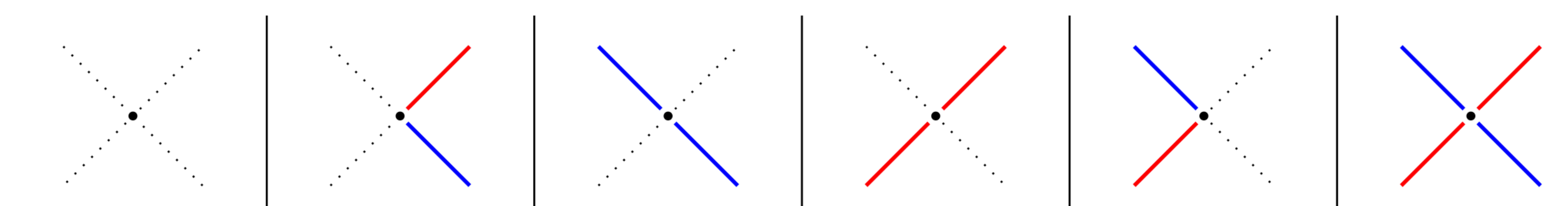
[P. 2020] Another form of the same determinant:

$$\frac{\prod_{i,j=1}^N(1-qu_iv_j)}{\prod_{1\le i<j\le N}(v_i-v_j)} \det\left[\frac{(1-\gamma)(q-\gamma^{-1}s_0^2)(1-u_iv_j)+(1-q)(1-\xi_0 s_0 u_i)(1-\xi_0^{-1}s_0v_j)}{(1-u_iv_j)(1-qu_iv_j)}\right]_{i,j=1}^N$$
$$= \det\Bigg[u_j^{N-i-1}\Bigg\{(1-s_0\xi_0u_j)\left(u_j-\frac{s_0}{\xi_0}\right)\prod_{l=1}^N\frac{1-qu_jv_l}{1-u_jv_l}$$
$$-\gamma^{-1}q^{N-i}(\gamma-s_0\xi_0u_j)\left(\gamma qu_j-\frac{s_0}{\xi_0}\right)\Bigg\}\Bigg]_{i,j=1}^N.$$

Inspired by **[Warnaar 2005]**, **[Cuenca 2017]**

Idea of proof of refined Cauchy





1	$\frac{(1-q)z}{1-z}$	$\frac{1-qz}{1-z}$	$\frac{1-qz}{1-z}$	$\frac{1-q}{1-z}$	q

Left-hand side is a $N \times N$ determinant identified through Lagrange interpolation

Approach goes back to 1980s work of Izergin and Korepin on six-vertex model with domain wall boundary conditions, and here essentially carries from **[Wheeler-Zinn Justin 2015]**

Case $s_0 = 0$ and connection to Macdonald averages

$$\prod_{i,j=1}^{N} \frac{1}{1-qu_iv_j} \sum_{\lambda\in\mathrm{Sign}_N} (\gamma q;q)_{m_0(\lambda)} \underbrace{\widetilde{\mathsf{F}}_\lambda(u_1,\ldots,u_N)\,\widetilde{\mathsf{F}}^*_\lambda(v_1,\ldots,v_N)}_{\text{functions at } s_0=0}$$

$$= \sum_{\lambda\in\mathrm{Sign}_N} \prod_{j=1}^{N} (1-(\gamma q)q^{\lambda_j+N-j})\, s_\lambda(u_1,\ldots,u_N)\, s_\lambda(v_1,\ldots,v_N)$$

$= \det[\cdots]$

$$\mathbb{E}_{\mathbf{sHL}(q)} (-\zeta;q)_{m_0(\lambda)} = \mathbb{E}_{\overbrace{\mathbf{MM}(q,q)}^{\text{Schur measure}}} \prod_{j=1}^{N} \left(1+\zeta q^{\lambda_j+N-j}\right)$$

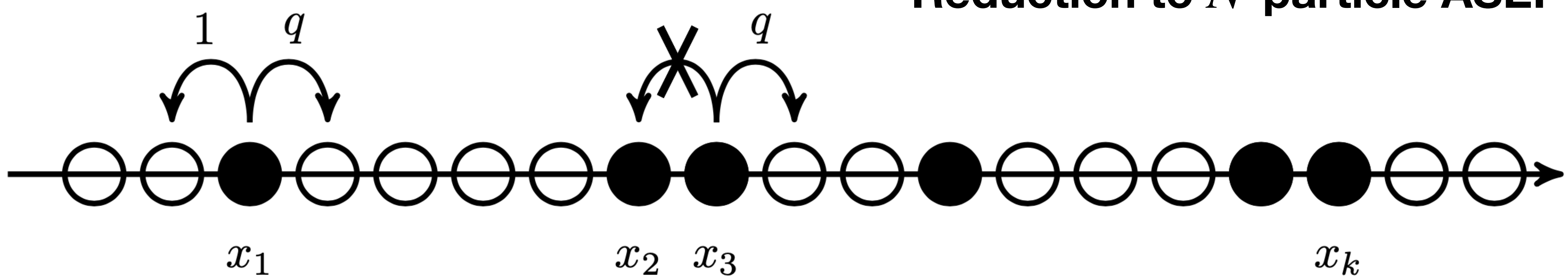
$$= \mathbb{E}_{\overbrace{\mathbf{MM}(\mathfrak{q},\mathfrak{t})}^{\text{Macdonald measure}}} \prod_{j=1}^{N} (1+\zeta \mathfrak{q}^{\lambda_j}\mathfrak{t}^{N-j})$$

//

Also related to the distribution of the height function of the **stochastic six-vertex model**

[Kirillov-Noumi 1999]
[Warnaar 2008]

Reduction to N-particle ASEP



The eigenfunctions of the Markov generator of ASEP are particular cases of the sHL's:

$$\Psi^r_{\vec{x}}(\vec{z}) = \sum_{\sigma \in S_N} \prod_{1 \le i < j \le N} \frac{z_{\sigma(i)} - q z_{\sigma(j)}}{z_{\sigma(i)} - z_{\sigma(j)}} \prod_{i=1}^{N} \left(\frac{1 - z_{\sigma(i)}}{1 - z_{\sigma(i)}/q} \right)^{-x_i}, \qquad \mathcal{A}\Psi^r_{\vec{x}}(\vec{z}) = \mathsf{ev}(\vec{z})\, \Psi^r_{\vec{x}}(\vec{z});$$

(Coordinate Bethe Ansatz)

$$\mathsf{ev}(\vec{z}) = -\sum_{j=1}^{N} \frac{(1-q)^2}{(1-z_j)(1-q/z_j)}.$$

[Tracy-Widom 2007] Transition function of ASEP is (all contours are around 1)

$$P_t(\vec{x} \to \vec{y}) = \frac{1}{N!(2\pi\mathbf{i})^N} \oint dz_1 \ldots \oint dz_N \, \frac{\prod_{i<j}(z_i - z_j)^2}{\prod_{i \ne j}(z_i - q z_j)}$$
$$\times \prod_{j=1}^{N} \frac{1 - 1/q}{(1-z_j)(1-z_j/q)} \; \exp\{t \cdot \mathsf{ev}(\vec{z})\}\, \Psi^r_{\vec{x}}(\vec{z})\, \Psi^{\ell}_{\vec{y}}(\vec{z}),$$

Summation identities for ASEP eigenfunctions

[Tracy-Widom 2007]

$$\sum_{0\le x_1<x_2<\ldots<x_N} \Psi^{\ell}_{\vec{x}}(\vec{z}) = \frac{(-q)^{\frac{N(N-1)}{2}}(1-z_1/q)\ldots(1-z_N/q)}{(1-1/q)^N z_1\ldots z_N}$$

(one of the "magic identities")

[Corwin-Liu 2019, unpublished], [P. 2020] derived from the sHL refined Cauchy:

$$\sum_{0\le x_1<x_2<\ldots<x_N} \Psi^{r}_{\vec{x}}(\vec{z})\, \Psi^{\ell}_{\vec{x}}(\vec{w})$$
$$= \prod_{j=1}^{N}(1-z_j)(1-w_j/q)\, \frac{(1/q-1)^{-N}\prod_{i,j=1}^{N}(z_i-qw_j)}{\prod_{1\le i<j\le N}(z_i-z_j)(w_j-w_i)} \det\left[\frac{1}{(z_i-w_j)(z_i-qw_j)}\right]_{i,j=1}^{N}$$

An example two-time quantity in ASEP:

$$\mathrm{Prob}\big(x_1(t_1)\ge k_1,\ x_1(t_2)\ge k_2\big) = \sum_{x_1'\ge k_1,\ x_1''\ge k_2} P_{t_1}(\vec{x}\to\vec{x}')\, P_{t_2-t_1}(\vec{x}'\to\vec{x}'')$$

[P. 2020]. For any initial condition $\vec{x}$. Can get arbitrary multitime formulas

$$\mathrm{Prob}(x_1(t_1) \geq k_1,\ x_1(t_2) \geq k_2) = \frac{(-1)^N q^{\frac{N(N-1)}{2}}}{(N!)^2(2\pi\mathbf{i})^{2N}} \oint \frac{dz_1}{1-z_1} \cdots \oint \frac{dz_N}{1-z_N} \oint \frac{dw_1}{w_1} \cdots \oint \frac{dw_N}{w_N}$$
$$\times \frac{\prod_{i<j}(z_i - z_j)(w_i - w_j) \prod_{i,j=1}^{N}(w_i - qz_j)}{\prod_{i \neq j}(z_i - qz_j)(w_i - qw_j)} \det\left[\frac{1}{(w_i - z_j)(w_i - qz_j)}\right]_{i,j=1}^{N}$$
$$\times \exp\{t_1\, \mathsf{ev}(\vec{z}) + (t_2 - t_1)\, \mathsf{ev}(\vec{w})\} \prod_{j=1}^{N} \left(\frac{1-z_j}{1-z_j/q}\right)^{k_1} \left(\frac{1-w_j}{1-w_j/q}\right)^{k_2-k_1} \Psi^r_{\vec{x}}(\vec{z}).$$

All integration contours are small positively oriented circles around 1, with $|z_i - 1| < |w_i - 1|$ for all z_i, w_j on the contours.

Asymptotics?

Single-time asymptotic analysis - **[Tracy-Widom 2008], …**

Multipoint analysis (six vertex model) - **[Dimitrov 2020]**

Determinantal models (TASEP):

- multipoint results for TASEP are available via Schur measures, as well as multitime results along **space-like** paths

- best general (multitime and multipoint) results are on a ring **[Baik, Liu 2016+]**;

- Multitime results on the line - **[Johansson, Rahman 2015+]**

Part 2

Spin Whittaker functions

j.w. Matteo Mucciconi

Spin Hall-Littlewood $\to$ spin q-Whittaker $\to$ spin Whittaker

fusion

limit $q \to 1$

Recall: **Whittaker symmetric functions** (Kostant, Givental, Bump, Stade, Gerasimov-Lebedev-Oblezin, Corwin-O'Connell-Seppalainen-Zygouras,...)

$u_{N,1} \cdots\cdots u_{N,N}$

$u_{N-1,1} \cdots u_{N-1,N-1}$

$\vdots$

$u_{1,1}$

$\lambda_j, u_j \in \mathbb{R}$

$$\underbrace{\psi_{\lambda_1,\ldots,\lambda_N}}_{\text{symm. in } \lambda_j}(\underline{u}_N) = \int_{\mathbb{R}^{N-1}} \psi_{\lambda_1,\ldots,\lambda_{N-1}}(\underline{u}_{N-1})\, Q_{\lambda_N}^{N\to N-1}(\underline{u}_N, \underline{u}_{N-1}) \prod_{k=1}^{N-1} du_{N-1,k},$$

where

$$Q_\lambda^{N\to N-1}(\underline{u}_N, \underline{u}_{N-1}) = e^{\mathrm{i}\lambda\left(\sum_{i=1}^N u_{N,i} - \sum_{i=1}^{N-1} u_{N-1,i}\right)} \prod_{i=1}^{N-1} \exp\left\{-e^{u_{N-1,i}-u_{N,i}} - e^{u_{N,i+1}-u_{N-1,i}}\right\}$$

[Mucciconi-P. 2020]

Spin Whittaker symmetric functions $\mathfrak{f}_{X_1,\ldots,X_N}(L_1,\ldots,L_N)$

Symmetric in $X_i \in \mathbb{R}$ and depend on $1 \le L_N \le \ldots \le L_1$, also on $S > 0$ with $|X_i| < S$

Reduction to the usual $\mathfrak{gl}_N$ Whittaker functions, $S \to +\infty$

$$L_i = S^{N+1-2i} e^{u_i}, \qquad X_k = -\mathrm{i}\lambda_k,$$

Conjecture (Mucciconi-P. 2020, holds modulo decay estimates**)**

$$\left(\frac{4\pi}{S 16^S}\right)^{\frac{N(N-1)}{4}} \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) \xrightarrow{S\to\infty} \psi_{\lambda_1,\ldots,\lambda_N}(u_1,\ldots,u_N)$$

“Combinatorial formula” (“spin Givental integral”)

$$\mathcal{A}_{S,X}(u,v,z) := \frac{1}{\mathrm{B}(S+X,S-X)}\left(1-\frac{v}{z}\right)^{S-X-1}\left(1-\frac{u}{v}\right)^{S+X-1}\left(1-\frac{u}{z}\right)^{1-2S}$$

$$\mathfrak{f}_X(\underline{L}_k;\underline{L}_{k+1}) := \mathbf{1}_{\underline{L}_k\prec\underline{L}_{k+1}}\left(\frac{L_{k+1,k+1}\cdots L_{k+1,1}}{L_{k,k}\cdots L_{k,1}}\right)^{-X}\prod_{i=1}^{k}\mathcal{A}_{S,X}(L_{k+1,i+1},L_{k,i},L_{k+1,i})$$

Interlacing $1 \le L_{k,k} \le L_{k-1,k-1} \le L_{k,k-1} \le \ldots \le L_{k-1,1} \le L_{k,1}$

$$\begin{array}{cccccc} L_N & L_{N-1} & \cdots & L_2 & & L_1 \\ & L'_{N-1} & \cdots & L'_2 & & L'_1 \\ & & L''_{N-2} & & & L''_1 \\ & & & \vdots & & \\ & & & L_1^{(N-1)} & & \end{array}$$

Definition. $$\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) := \int_{\underline{L}_{N-1}\prec\underline{L}_N}\mathfrak{f}_{X_1,\ldots,X_{N-1}}(\underline{L}_{N-1})\,\mathfrak{f}_{X_N}(\underline{L}_{N-1};\underline{L}_N)\frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}}$$

Examples. $$\mathfrak{f}_{X_1}(L_{1,1}) = L_{1,1}^{-X_1}$$

$$\mathfrak{f}_{X,Y}(u,z) = (z/u)^S u^{-X-Y}\,{}_2F_1\left(\begin{matrix}S+X\,,\;S+Y\\2S\end{matrix}\,\middle|\,1-\frac{z}{u}\right).$$

“Dual” functions. $$\mathfrak{g}_Y(\underline{\widetilde{L}}_k;\underline{L}_k) = \frac{L_{k,1}^{-Y}}{\Gamma(S-Y)}\left(1-\frac{\widetilde{L}_{k,1}}{L_{k,1}}\right)^{S-Y-1}\mathfrak{f}_{-Y}(\underline{\ell}_{k-1};\underline{\widetilde{L}}_k),$$

$$\mathfrak{g}_{Y_1,\ldots,Y_M}(\underline{L}_N) = \begin{cases}\displaystyle\int\mathfrak{g}_{Y_1,\ldots,Y_{M-1}}(\underline{\widetilde{L}}_N)\mathfrak{g}_{Y_M}(\underline{\widetilde{L}}_N;\underline{L}_N)\frac{d\underline{\widetilde{L}}_N}{\underline{\widetilde{L}}_N} & \text{if } N<M,\\[2ex] \displaystyle\int\mathfrak{g}_{Y_1,\ldots,Y_{N-1}}(\underline{\widetilde{L}}_{N-1})\mathfrak{g}_{Y_N}(\underline{\widetilde{L}}_{N-1};\underline{L}_N)\frac{d\underline{\widetilde{L}}_{N-1}}{\underline{\widetilde{L}}_{N-1}} & \text{if } N=M.\end{cases}$$

$$\mathfrak{g}_Y(L) = \mathfrak{g}_Y(1;L) = \frac{L^{-Y}(1-L^{-1})^{S-Y-1}}{\Gamma(S-Y)}.$$

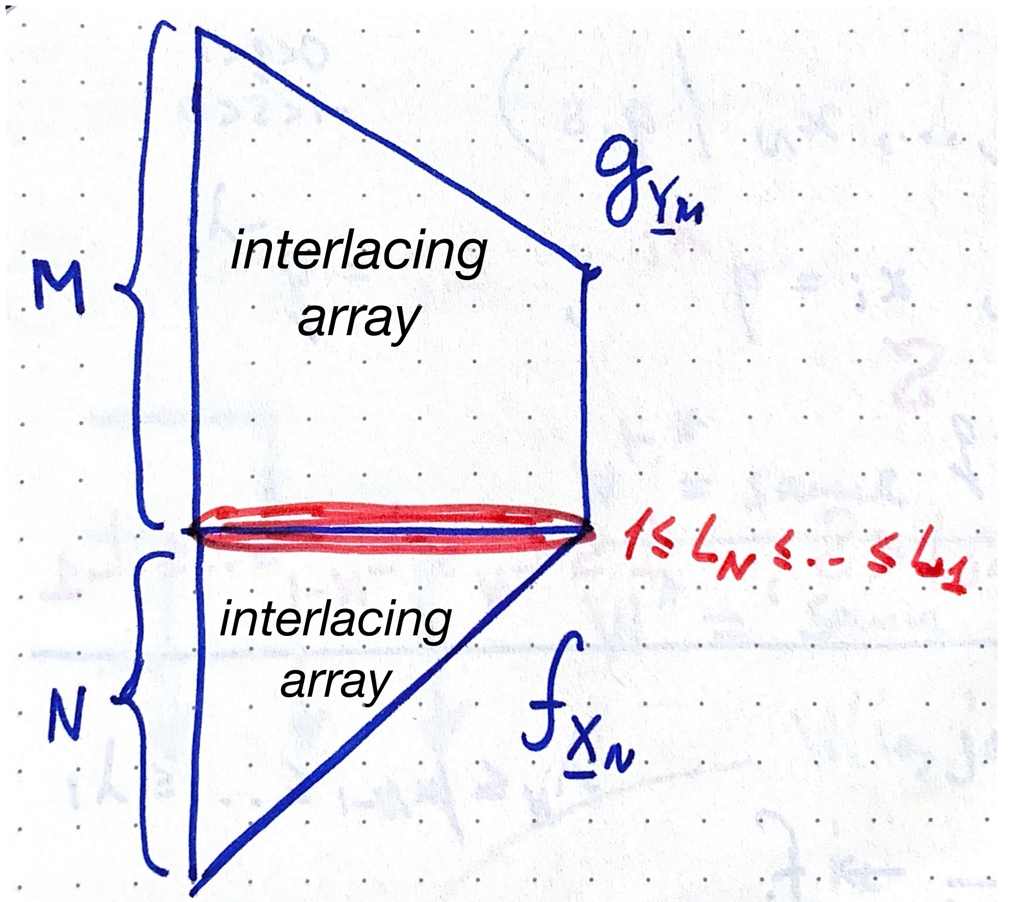
Properties

(1) $\mathfrak{f}_{X_1,\ldots,X_N}(a\underline{L}_N) = a^{-X_1-\ldots-X_N}\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N), \qquad a > 1.$

(2) Cauchy identity, $M \geq N$

$$\int \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N)\,\mathfrak{g}_{Y_1,\ldots,Y_M}(\underline{L}_N)\,\frac{d\underline{L}_N}{\underline{L}_N} = \prod_{j=1}^{M}\frac{\Gamma(X_1+Y_j)}{\Gamma(S+X_1)}\left(\prod_{i=2}^{N}\frac{\Gamma(X_i+Y_j)\Gamma(2S)}{\Gamma(S+X_i)\Gamma(S+Y_j)}\right).$$

Generalizes **[Bump-Stade 2002, Corwin-O'Connell-Seppalainen-Zygouras 2011]** and reduces to these as $S \to +\infty$



(3) Difference eigenoperators ($\mathcal{T}_X$ - shift by 1) like the Macdonald ones. But *only* 2, not N

$$\mathfrak{D}_1 := \sum_{i=1}^{N}\prod_{\substack{j=1\\j\neq i}}^{N}\frac{X_i+S}{X_i-X_j}\mathcal{T}_{X_i}, \qquad \overline{\mathfrak{D}}_1 := \sum_{i=1}^{N}\prod_{\substack{j=1\\j\neq i}}^{N}\frac{X_i-S}{X_i-X_j}\mathcal{T}_{X_i}^{-1}.$$

$$\mathfrak{D}_1\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) = L_{N,N}^{-1}\,\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N),$$
$$\overline{\mathfrak{D}}_1\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) = L_{N,1}\,\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N).$$

(4) Deformed quantum Toda (scaling limit of Pieri rules, similar to **[Gerasimov-Lebedev-Oblezin 2011-12]**)

$$\mathcal{H}_2 := -\frac{1}{2}\sum_{i=1}^{N}\partial_{u_i}^2 + \sum_{1\le i<j\le N} S^{-2(j-i)} e^{u_j-u_i}(S-\partial_{u_i})(S+\partial_{u_j}).$$

Additive variables u_i

$$L_i = S^{N+1-2i}e^{u_i}.$$

Theorem.

$$\mathcal{H}_2 \mathfrak{f}_{\underline{X}}^{add}(u_1,\ldots,u_N) = -\frac{1}{2}\left(X_1^2+\cdots+X_N^2\right)\mathfrak{f}_{\underline{X}}^{add}(u_1,\ldots,u_N).$$

Remark. For $S \to +\infty$ we get the usual $\mathfrak{gl}_N$ quantum Toda Hamiltonian

$$\mathcal{H}_2^{\mathrm{Toda}} := -\frac{1}{2}\sum_{i=1}^{N}\partial_{u_i}^2 + \sum_{i=1}^{N-1} e^{u_{i+1}-u_i}.$$

Simple roots vs positive roots ?

(5) Conjectural "weak" orthogonality with "spin Sklyanin measure"

$$\int_{(\mathrm{i}\mathbb{R})^N} \mathfrak{f}_{\underline{Z}}(\underline{L}_N)\,\mathfrak{f}_{-\underline{Z}}(\underline{L}'_N)\,\mathfrak{M}_S^N(\underline{Z})\,dZ_1\ldots dZ_N = \prod_{i=1}^{N-1}\left(1-\frac{L_{N,i+1}}{L_{N,i}}\right)^{1-2S}\delta_{\underline{L}_N-\underline{L}'_N},$$

$$\mathfrak{M}_S^N(\underline{Z}) = \frac{1}{N!(2\pi\mathrm{i})^N}\prod_{1\le i\ne j\le N}\frac{\Gamma(S+Z_i)\Gamma(S-Z_i)}{\Gamma(2S)\Gamma(Z_i-Z_j)},$$

Spin Whittaker processes: Application to probability

Define a *probability measure* based on the spin Whittaker functions

(like Schur or Macdonald processes)

$$\mathfrak{P}_{\mathbf{X};\mathbf{Y}}(\underline{\underline{L}}_N) = \frac{\mathfrak{f}_{X_1}(\underline{L}_1)\mathfrak{f}_{X_2}(\underline{L}_1;\underline{L}_2)\cdots\mathfrak{f}_{X_N}(\underline{L}_{N-1};\underline{L}_N)\mathfrak{g}_{\mathbf{Y}}(\underline{L}_N)}{\Pi(\mathbf{X};\mathbf{Y})}.$$

$$\Pi(\mathbf{X};\mathbf{Y}) = \prod_{j=1}^{T} \frac{\Gamma(X_1+Y_j)}{\Gamma(S+X_1)} \left(\prod_{i=2}^{N} \frac{\Gamma(X_i+Y_j)\Gamma(2S)}{\Gamma(S+X_i)\Gamma(S+Y_j)} \right).$$

$$\mathbf{X} = (X_1, \ldots, X_N)$$

$$\mathbf{Y} = (Y_1, \ldots, Y_T)$$

We also have Markov dynamics on spin Whittaker processes which increase the parameter T.

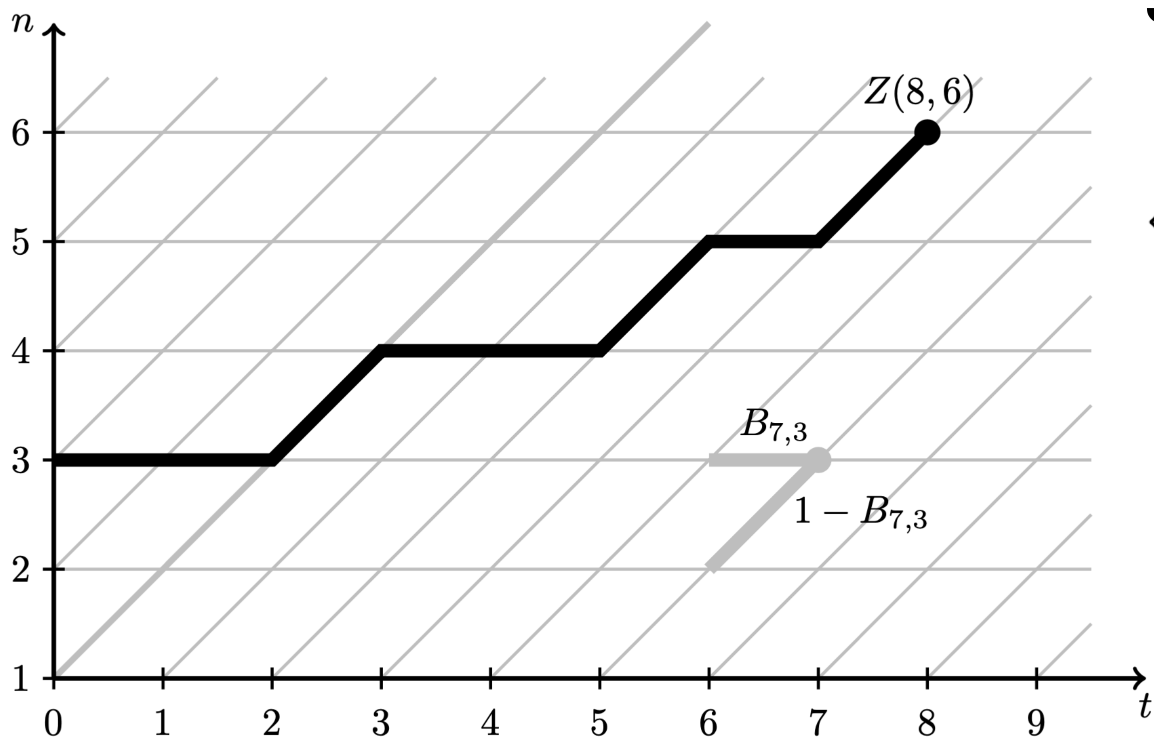
Theorem (Mucciconi-P. 2020).
The marginals $L_{k,k}(T)^{-1}$ have the same distribution as the strict-weak beta polymer model $Z(k,T)$ of [Barraquand-Corwin 2015].

The marginals $L_{k,1}(T)^{-1}$ have the same distribution as the “weird” beta polymer model $\tilde{Z}(k,T)$ of [Corwin-Matveev-P. 2018].

Both polymer models arise as $q \to 1$ limits of q-Hahn particle systems of **[Povolotsky 2013, Corwin 2014, Corwin-Matveev-Petrov 2018]**

The “weird” model as $S \to \infty$ also reduces to the more usual **log-gamma** polymer model.
The strict-weak beta polymer reduces to the strict-weak log-gamma polymer.

Beta polymers



Strict-weak beta polymer

$$\begin{cases} Z(i,j) = Z(i,j-1)B_{i,j} + Z(i-1,j-1)(1-B_{i,j}) & \text{for } 1 < i \le j; \\ Z(1,j) = Z(1,j-1)B_{1,j} & \text{for } j > 0; \\ Z(i,0) = 1 & \text{for } i > 0. \end{cases}$$

$$B_{i,j} \sim \mathcal{B}eta(X_i + Y_j, S - Y_j)$$

$$\mathcal{B}(m,n)[x] = \frac{x^{m-1}(1-x)^{n-1}}{\mathrm{B}(n,m)} \qquad \text{for } x \in (0,1).$$

"Weird" beta polymer-type model - a random recursion with cases

$$\widetilde{Z}(i,j) = \begin{cases} 1 & \text{for } j = 0, \\ \widetilde{Z}(1,j-1)\widetilde{B}_{1,j} & \text{for } i = 1, \\ W^{>}_{i,j}\widetilde{Z}(i,j-1) + (1-W^{>}_{i,j})\widetilde{Z}(i-1,j) & \text{if } \widetilde{Z}(i,j-1) > \widetilde{Z}(i-1,j), \\ (1-W^{<}_{i,j})\widetilde{Z}(i,j-1) + W^{<}_{i,j}\widetilde{Z}(i-1,j) & \text{if } \widetilde{Z}(i,j-1) < \widetilde{Z}(i-1,j), \end{cases}$$

$$\widetilde{B}_{1,j} \sim \mathcal{B}eta^{-1}(X_1 + Y_j, S - Y_j)$$

Difference operators
$\Rightarrow$ ∮ formulas

$$W^{>}_{i,j} \sim \mathcal{NBB}^{-1}\left(2S-1, \frac{\widetilde{Z}(i-1,j) - \widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i,j-1) - \widetilde{Z}(i-1,j-1)}, X_i + Y_j, S - Y_j\right),$$

$$W^{<}_{i,j} \sim \mathcal{NBB}^{-1}\left(2S-1, \frac{\widetilde{Z}(i,j-1) - \widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i-1,j) - \widetilde{Z}(i-1,j-1)}, X_i + Y_j, S - X_i\right).$$

Where NBB is a random variable on [0,1] with density

$$\mathcal{NBB}(r,p,m,n)[x] = \frac{(1-p)^r x^{m-1}(1-x)^{n-1}}{\mathrm{B}(n,m)}\, {}_2F_1\left(\begin{matrix} r, n+m \\ n \end{matrix} \,\middle|\, p(1-x)\right),$$

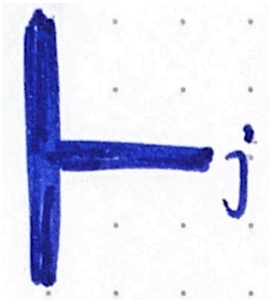
Part 2a

Spin q-Whittaker polynomials

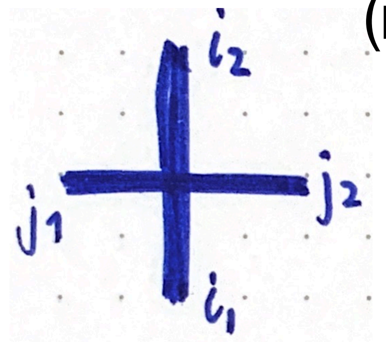
j.w. Matteo Mucciconi

Spin q-Whittaker weights = fusion of the spin Hall-Littlewood ones

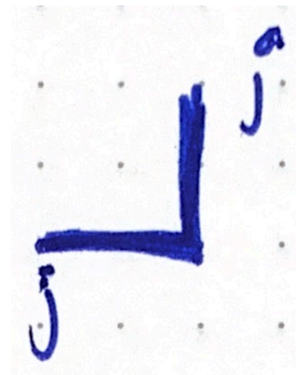
(most weights become polynomial in x or y)



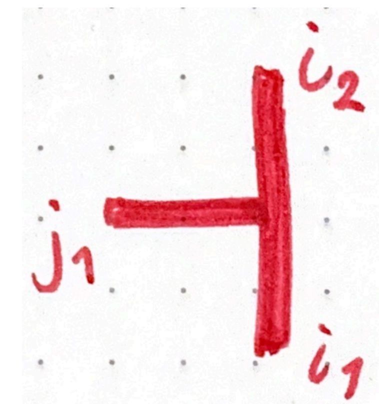
$$x^j \, \frac{(-s/x;q)_j}{(q;q)_j}$$



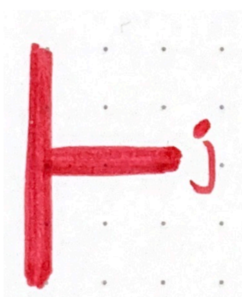
$$\mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\ge j_2}\, x^{j_2}\,\frac{(-s/x;q)_{j_2}(-sx;q)_{i_1-j_2}(q;q)_{i_2}}{(q;q)_{j_2}(q;q)_{i_1-j_2}(s^2;q)_{i_2}}$$



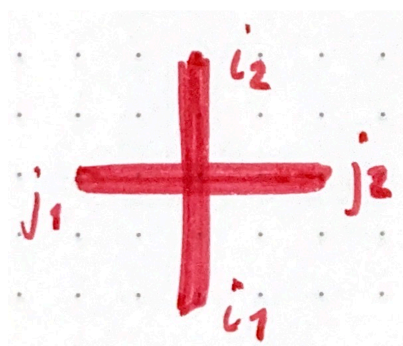
$$\frac{(q;q)_j}{(-s/x;q)_j}.$$



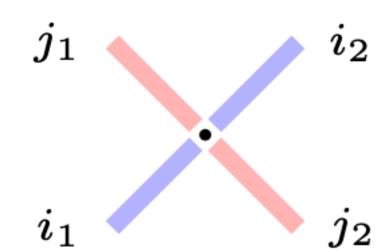
$$\mathbf{1}_{i_1=j_1+i_2}$$



$$x^j \, \frac{(-s/x;q)_j}{(q;q)_j}$$



$$\mathbf{1}_{i_2+j_1=i_1+j_2}\mathbf{1}_{i_2\ge j_2}\,\frac{y^{j_2}(q;q)_{i_2}(-s/y;q)_{j_2}(-sy;q)_{i_2-j_2}}{(q;q)_{i_2-j_2}(q;q)_{j_2}(s^2;q)_{i_2}}$$



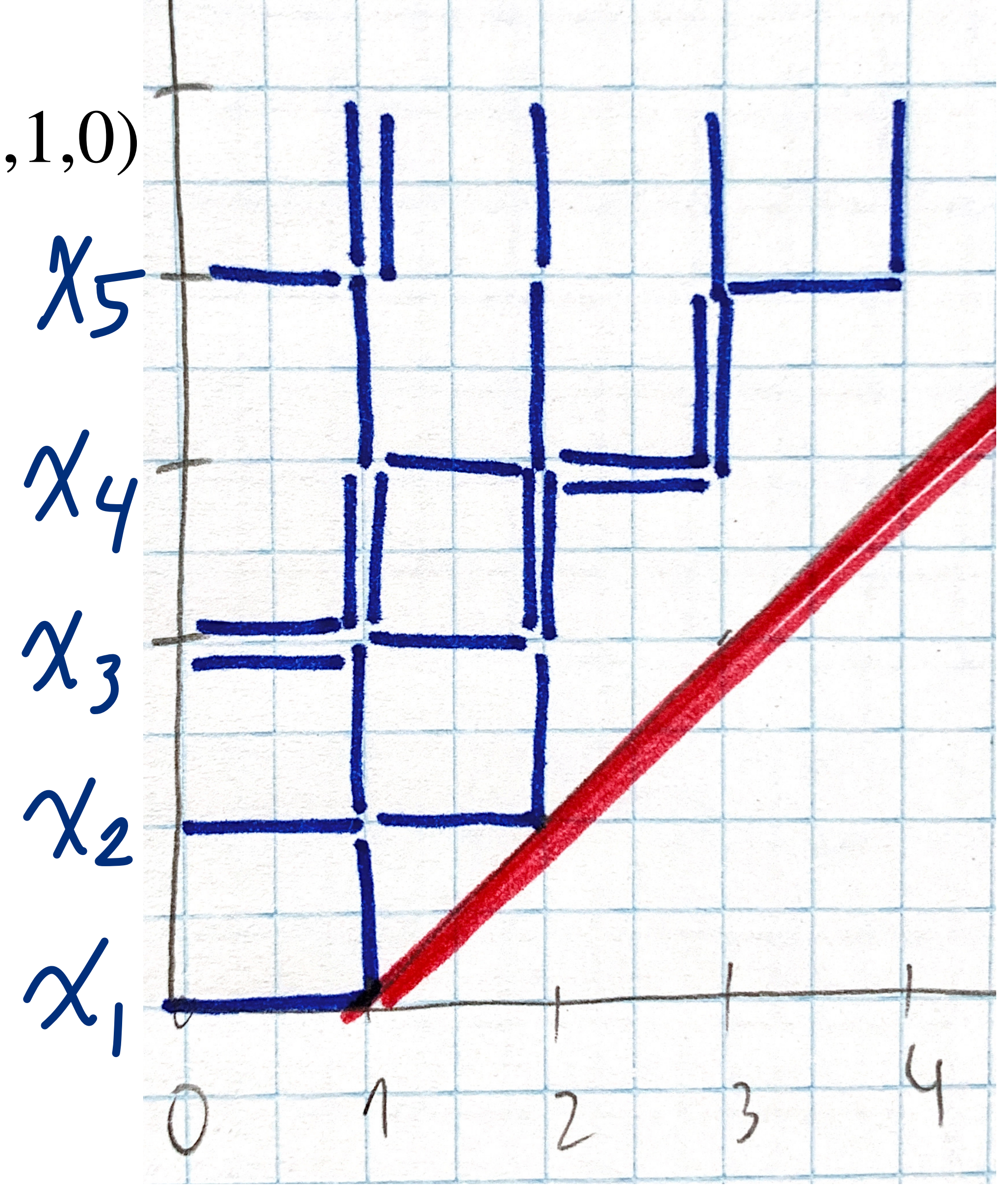
$$\mathbb{R}_{x,y,s}(i_1,j_1;i_2,j_2) := \mathbf{1}_{i_2+j_1=i_1+j_2}\,\frac{q^{i_2 i_1+\frac{1}{2}j_2(j_2-1)}(sx)^{j_2}(q;q)_{j_1}}{(s^2;q)_{j_1+i_2}(q;q)_{j_2}(q;q)_{i_2}(-q/(sx);q)_{i_1-j_1}}$$

$$\times\, {}_4\overline{\phi}_3\left(\begin{matrix} q^{-i_2}; q^{-i_1}, -sy, -q/(sx) \\ -s/x, q^{1+j_2-i_2}, -yq^{1-i_1-j_2}/s\end{matrix}\ \Big|\, q,q\right).$$

$${}_4\overline{\phi}_3\left(\begin{matrix} q^{-n} & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3\end{matrix}\,; q, z\right) = \sum_{j=0}^{n} z^j \frac{(q^{-n};q)_j}{(q;q)_j}(a_1,a_2,a_3;q)_j\,(q^j b_1, q^j b_2, q^j b_3;q)_{n-j}.$$

Spin q-Whittaker symmetric polynomials

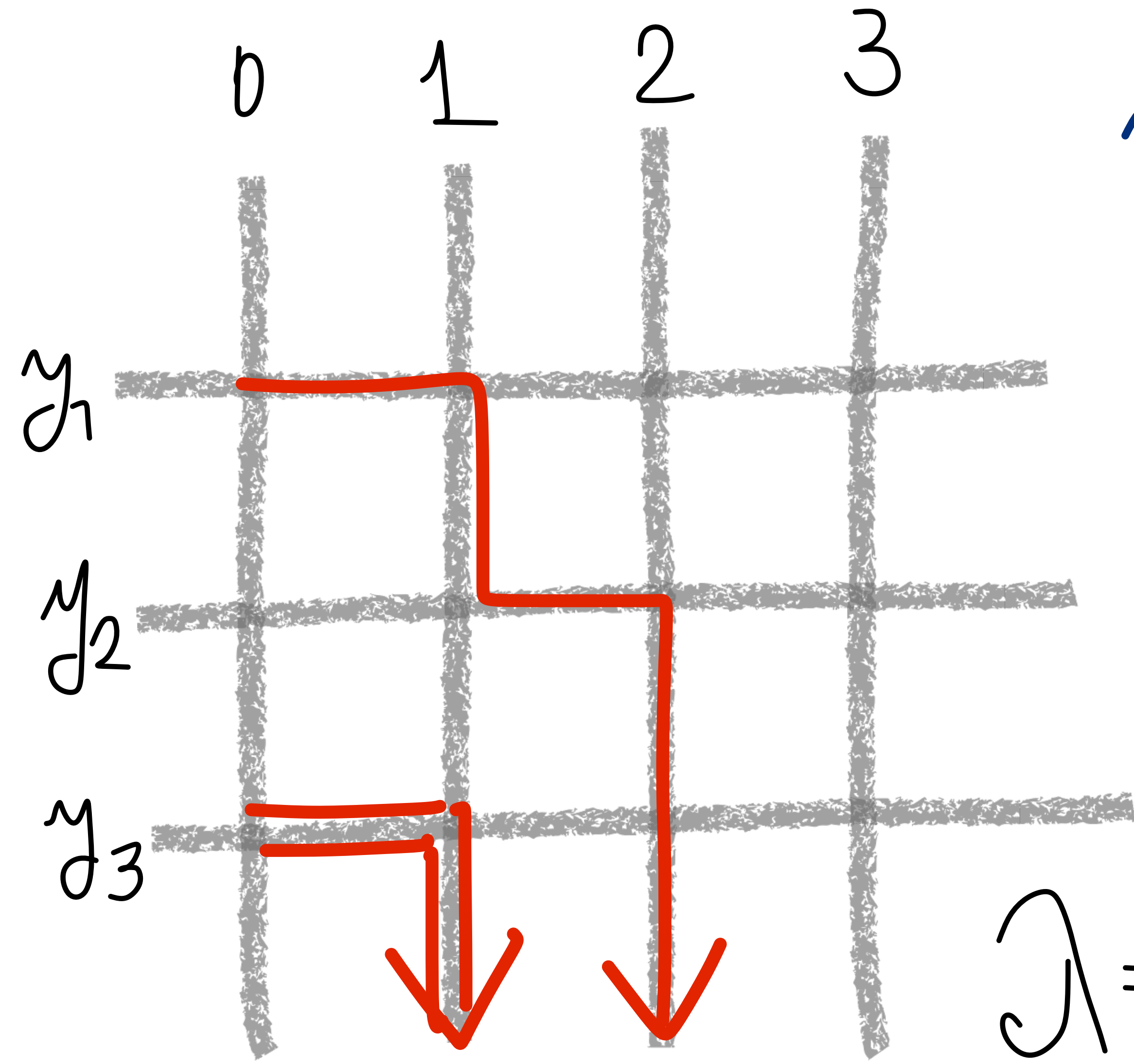
$\lambda = (5,3,2,1,0)$



$\mathbb{F}_\lambda(x_1, \ldots, x_N)$

$\lambda_i - \lambda_{i+1} = \#\{$paths through the i-th vertical edge$\}$.

$\mathbb{F}^*_\lambda(y_1, \ldots, y_k)$



$\lambda = (3,1,0)$

Spin q-Whittaker polynomials

$$\mathbb{F}_{\lambda/\mu}(x) := x^{|\lambda|-|\mu|} \prod_{i=1}^{k} \frac{(-s/x;q)_{\lambda_i-\mu_i}(-sx;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\lambda_i-\lambda_{i+i}}}$$

(*)

(this is a polynomial in x)

$$0 \le \lambda_{k+1} \le \mu_k \le \lambda_k \le \dots \mu_1 \le \lambda_1, \qquad \lambda_i, \mu_i \in \mathbb{Z}$$

$$\mathbb{F}_{\nu}(x_1,\dots,x_n) = \sum_{\varkappa} \mathbb{F}_{\varkappa}(x_1,\dots,x_{n-1})\, \mathbb{F}_{\nu/\varkappa}(x_n)$$

This is a symmetric polynomial in $x_1, \dots, x_n$, which follows from YBE

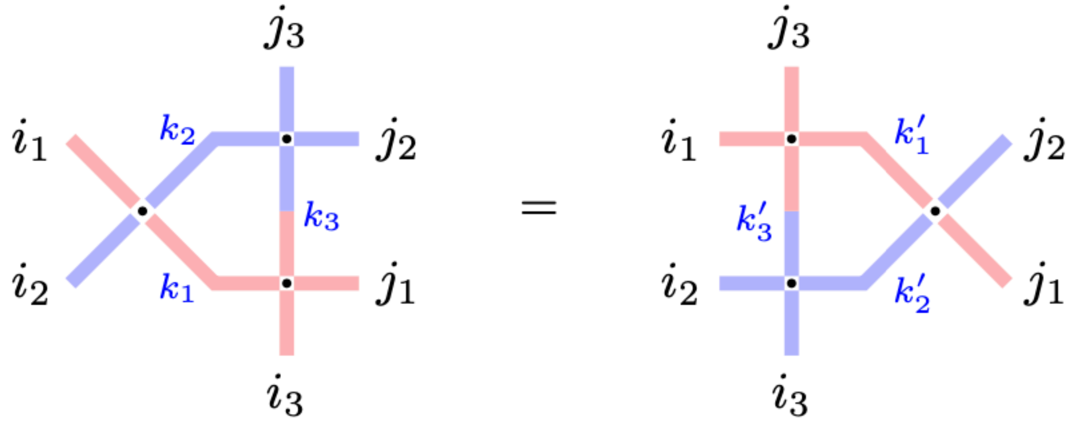
Borodin-Wheeler's version (2017)

$$\mathbb{F}^{BW}_{\lambda/\mu}(x) = \frac{(-s/x;q)_{\lambda_{k+1}}}{(s^2;q)_{\lambda_{k+1}}}\, \mathbb{F}_{\lambda/\mu}(x).$$

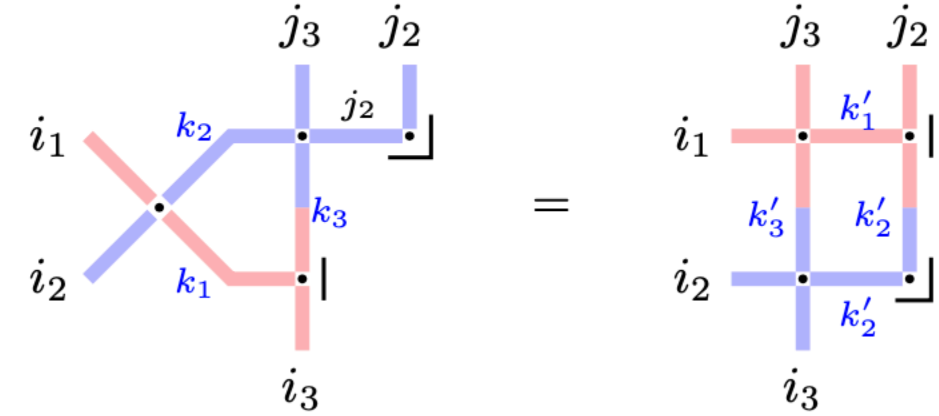
$$\mathbb{F}_{\lambda}(0,x_2,\dots,x_n) = \mathbb{F}^{BW}_{\lambda}(x_2,\dots,x_n).$$

We made a typo implementing $\mathbb{F}^{BW}_{\lambda/\mu}$ and wrote **(*)** instead of the correct expression. But surprisingly **(*)** leads to symmetric polynomials satisfying *nicer* properties - that is how our sqW polynomials were discovered. 😁

Yang-Baxter equation example ($q = s = 0$)

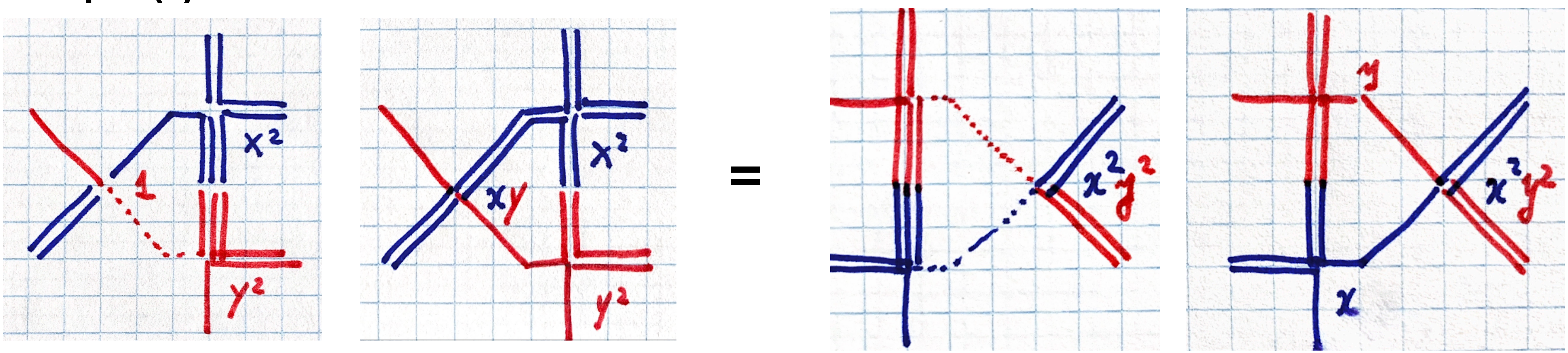


(a)

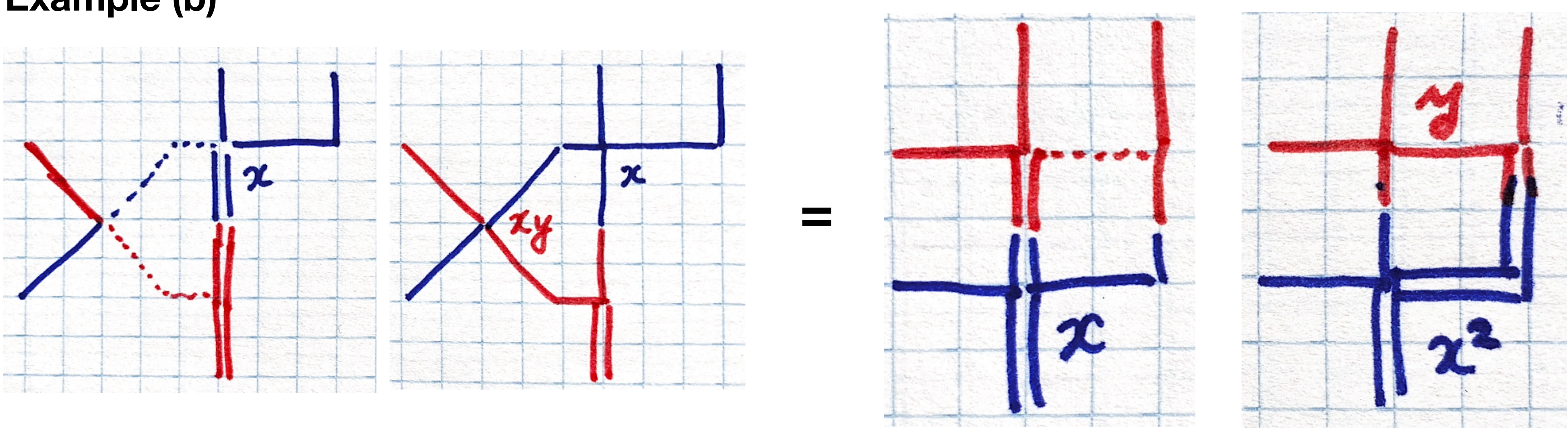


(b)

Example (a)



Example (b)



Cauchy identity

$$\sum_{\lambda \in \mathrm{Sign}_N} \mathbb{F}_\lambda(x_1, \ldots, x_N)\, \mathbb{F}^*_\lambda(y_1, \ldots, y_k) = \prod_{j=1}^{k} \left(\frac{(-sy_j; q)_\infty}{(s^2; q)_\infty} \right)^{N-1} \prod_{i=1}^{N} \prod_{j=1}^{k} \frac{(-sx_i; q)_\infty}{(x_i y_j; q)_\infty}.$$

q-difference operators ($T_{q,x}$ maps $f(x)$ to $f(qx)$)

$$\mathfrak{D}_1 := \sum_{i=1}^{N} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{(1 + sx_i)}{1 - x_i/x_j} T_{q,x_i}, \qquad \mathfrak{D}_1 \mathbb{F}_\lambda(x_1, \ldots, x_N) = q^{\lambda_N} \mathbb{F}_\lambda(x_1, \ldots, x_N).$$

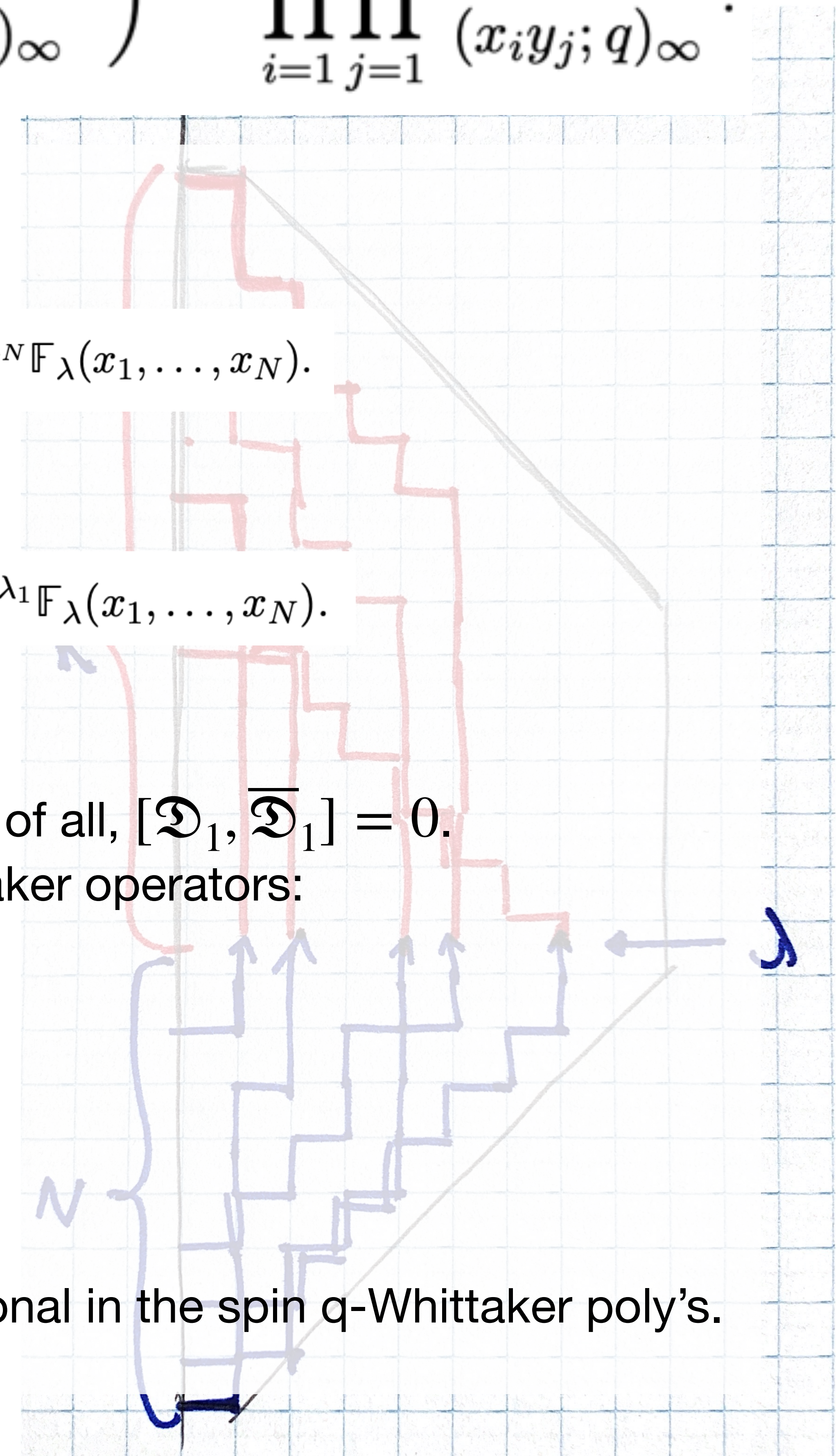
$$\overline{\mathfrak{D}}_1 := \sum_{i=1}^{N} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{(1 + s/x_i)}{1 - x_j/x_i} T_{q^{-1},x_i}. \qquad \overline{\mathfrak{D}}_1 \mathbb{F}_\lambda(x_1, \ldots, x_N) = q^{-\lambda_1} \mathbb{F}_\lambda(x_1, \ldots, x_N).$$

In the spin deformation the situation is more mysterious. First of all, $[\mathfrak{D}_1, \overline{\mathfrak{D}}_1] = 0$.
Next, both of them are conjugations of the first order q-Whittaker operators:

$$\mathbb{U}_N := \prod_{i=1}^{N} \frac{1}{(-sx_i; q)_\infty^{N-1}}, \qquad \mathbb{V}_N := \prod_{i=1}^{N} \frac{1}{(-s/x_i; q)_\infty^{N-1}}.$$

$$\mathfrak{D}_1 = \mathbb{U}_N^{-1} W_N^1 \mathbb{U}_N, \qquad \overline{\mathfrak{D}}_1 = \mathbb{V}_N^{-1} \tilde{W}_N^1 \mathbb{V}_N,$$

Same conjugations of the q-Whittaker operators are not diagonal in the spin q-Whittaker poly's.



Limit transitions sqW → sW → W

$$\mathbb{F}_\lambda(x_1,\ldots,x_N \,|\, q,s) \qquad \begin{matrix} 0<q<1 \\ -1<s<0 \end{matrix}$$

$$q\to 1, \quad x_i = q^{X_i}, \quad L_i = q^{-\lambda_i}$$

$$s = -q^{S}$$

$$S>0, \quad |X_i|<S, \quad 1\le L_N \le L_{N-1} \le \ldots \le L_1$$

Theorem (Mucciconi-P. 2020**)**

$$\lim_{q\to 1}\frac{\mathbb{F}_\lambda(x_1,\ldots,x_N)}{(-\log q)^{N(N-1)/2}} = \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N)$$

$$1\le L_N \le \ldots \le L_1$$

Here $\mathfrak{f}_{X_1,\ldots,X_N}$ is the *spin Whittaker* function, which is symmetric in X_i, depends on $\underline{L}_N$ and on a parameter S.

Reduction to the usual $\mathfrak{gl}_N$ Whittaker functions, $S\to+\infty$

$$L_i = S^{N+1-2i}e^{u_i}, \qquad X_k = -\mathrm{i}\lambda_k,$$

Conjecture (Mucciconi-P. 2020, holds modulo decay estimates**)**

$$\left(\frac{4\pi}{S16^S}\right)^{\frac{N(N-1)}{4}} \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) \xrightarrow{S\to\infty} \psi_{\lambda_1,\ldots,\lambda_N}(u_1,\ldots,u_N)$$

Conclusions and further problems

- New identities relating Izergin-Korepin type determinants and expectations.
- A multitime ASEP formula; asymptotics unclear
- Our initial motivation is in probability, and we have added an extra parameter to q-Whittaker / Whittaker setup ([COSZ], [BC] 2010+), building symmetric functions for beta random polymers
- How to prove the conjectural orthogonalities?
- Are there higher order eigenoperators for sqW or sW, like for the Macdonald polynomials?
- Polymer interpretation of multilayer distributions? Multilayer beta polymers? “Geometric RSK” for beta polymers and spin Whittaker processes?
- Representation theory / number theory behind spin Whittaker functions?
- Other symmetry types?