Refined dual Grothendieck polynomials from integrability

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Joint work with Kohei Motegi

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Schubert calculus

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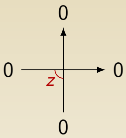
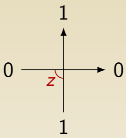
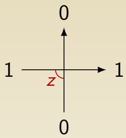
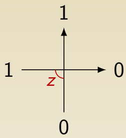
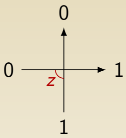
- The Grassmannian: k-dimensional planes in $\mathbb{C}^n$.
- Basis for cohomology using Schubert varieties.
- Represented by Schur functions $s_\lambda(\mathbf{x})$ such that λ inside a $k \times (n-k)$ rectangle.
- Many well-known formulas, including sum over semistandard tableaux, the Jacobi–Trudi formula, and using 5-vertex integrable lattice models.

5-vertex model for Schur functions

a_1	b_1	b_2	c_1	c_2
				
1	1	z	z	1

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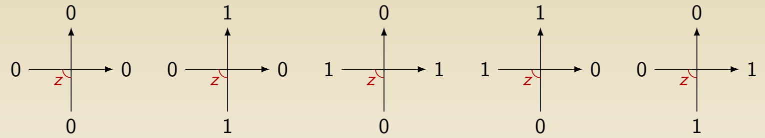
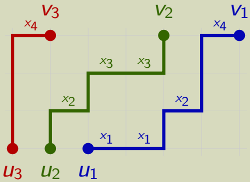


a_1	b_1	b_2	c_1	c_2
1	1	z	z	1

Example

1	1	2	4
2	3	3	
4			

$\longleftrightarrow$



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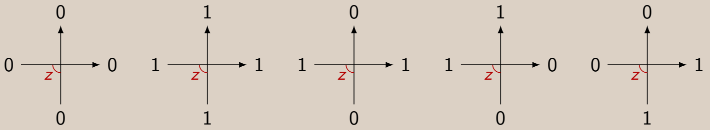
Theorem (Motegi–Sakai, 2013)

The 5-vertex model with L-matrix



a_1	a_2	b_2	c_1	c_2
1	1	z	1	$1+\beta z$

is integrable and the partition function is $G_\lambda(\mathbf{x};\beta)$.

Dual Grothendieck polynomials

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Theorem (Lam–Pylyavskyy, 2008)

There exists bijection between RPPs and a semistandard tableau (P, E) of shape μ and λ/μ such that entries in row i of E are at most i. The tableau E is called an elegant tableau.

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Definition

The *refined dual Grothendieck polynomial* is

$$g_\lambda(\mathbf{x};\mathbf{t}) = \sum_{\mu \subseteq \lambda} e^{\mu}\lambda(\mathbf{t}) s_\mu(\mathbf{x}),$$

where $e^{\mu}\lambda(\mathbf{t}) = \sum_E \mathbf{t}^{\mathrm{wt}(E)}$ is over elegant tableau E of shape λ/μ.

Outline

1. Refined dual Grothendiecks
2. The lattice model
   - Nonintersecting lattice paths
   - Combinatorial interpretation
3. Results

Flagged Schur functions

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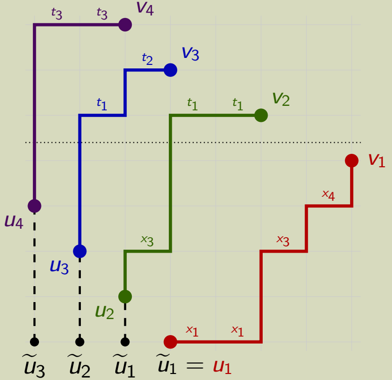
Theorem

The partition function of this jagged 5-vertex is a refined dual Grothendieck polynomial.

Lattice path construction

Example

Let $n = 5$, $\lambda = 4322$, and $\mu = 41$. The shaded portion below is the elegant tableau.



$\longmapsto$

1	1	3	4
3	1	1	
1	2		
3	3		

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Proof.

Notice that the jagged portion precisely corresponds to the elegant tableau. □

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Corollary

The refined dual Grothendiecks are given by Lascoux's multi-Schur functions, and are specializations of certain Schubert polynomials (and Demazure characters/key polynomials).

Outline

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2. The lattice model
3. Results
   - Identities
   - Symmetries
   - Further directions

Jacobi–Trudi formulas

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$$g_\lambda(\mathbf{x};\mathbf{t}) = \det\left[e_{\lambda'_i+j-i}(\mathbf{x}, t_1, \ldots, t_{\lambda'_i-1})\right]_{i,j=1}^{n}$$

Cauchy-type identity

Corollary

Let $\lambda^\dagger$ *be the complement of* λ *in* m^ℓ *and* $\mathbf{t}^\dagger = (t_{\ell-1}, \ldots, t_1)$*:*

$$s_{m^\ell}(\mathbf{x}, \mathbf{t}, \mathbf{y}) = \sum_{\lambda \subseteq m^\ell} g_\lambda(\mathbf{x}; \mathbf{t}) g_{\lambda^\dagger}(\mathbf{y}; \mathbf{t}^\dagger).$$

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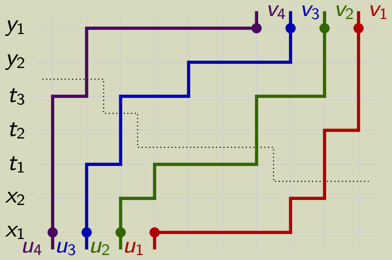

$\mapsto$

x_1	x_1	x_1	x_1	x_2
x_2	t_1	t_1	t_1	
t_1				
t_3				

,

y_1	y_1	y_1	y_1	y_1
y_2	y_2	y_2	t_3	t_3
t_3	t_3			
t_2				

where $y_1 < y_2 < t_3 < t_2 < t_1$.

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Corollary (Branching rule)

$$g_\lambda(\mathbf{x}, \gamma; \mathbf{t}) = \sum_{\mu \subseteq \lambda} \gamma^{\lambda_1 - \mu_1} t_1^{\lambda_2 - \mu_2} \cdots t_{\ell-1}^{\lambda_\ell - \mu_\ell} g_\mu(\mathbf{x}; \gamma, \mathbf{t})$$

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Corollary

$$s_\nu(\mathbf{x},\widetilde{\mathbf{t}}) = \sum_{\lambda\subseteq\mu} p_\nu^\lambda(\widetilde{\mathbf{t}}) g_\lambda(\mathbf{x};\mathbf{t}),$$

where $\widetilde{\mathbf{t}} = (t_1,\ldots,t_m)$ *and* $p_\nu^\lambda(\widetilde{\mathbf{t}})$ *are semistandard skew tableau of shape* λ/μ *with max entry* m *and all entries in row* i *being at least* i*.*

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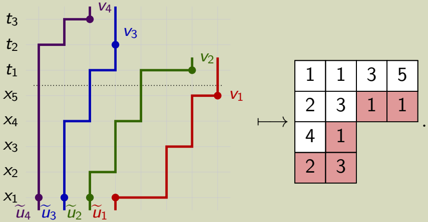

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Corollary (Yeliussizov, 2019, Thm 5.2(iv))

$$\sum_{\ell(\lambda) \le \ell} \prod_{i=1}^{\ell} t_i^{-\lambda_i} g_\lambda(\mathbf{x};\mathbf{t}) = \prod_{i=1}^{n} \prod_{j=1}^{\ell} \frac{1}{1 - t_j^{-1} x_i} = \prod_{i=1}^{n} \prod_{j=1}^{\ell} \frac{t_j}{t_j - x_i}.$$

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Corollary

$$g_\lambda(\mathbf{x}; \beta) = \frac{1}{(2\pi i)^\ell} \oint \cdots \oint \prod_{i<j} \frac{(z_j - z_i)(1 - \beta z_j - \beta z_i)}{(1 - \beta z_j)} \frac{\prod_{i=1}^{\ell} z_i^{\lambda_i + \ell - i}}{\prod_{i,m=1}^{\ell}(z_i - x_m)} \, dz_1 \cdots dz_\ell$$

Thank you!

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